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Comprehensive bifurcation analysis in a delayed epidemic model with media coverage and asymptomatic infection via fractional PD control*

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Abstract. It is well known that both media coverage and the presence of asymptomatic patients have a significant impact on the spread and control of infectious diseases. Therefore, this paper proposes a class of fractional epidemic models that incorporate media coverage and asymptomatic infection. Also, the time delay for individuals' response to the current media coverage, as well as the time delay of media coverage, are incorporated in our proposed model to make it more practical. Based on fractional proportional-derivative (PD) control method, Hopf bifurcation is investigated by taking the sum of the two time delays and the order of the fractional derivatives as bifurcation parameters, respectively. Some sufficient delayed-induced and order-induced bifurcation conditions are given. The application and effectiveness of the presented theoretical results are illustrated through a simulation example. Furthermore, the impact of the feedback control gains and the media effect weight on the stability and Hopf bifurcation of the considered epidemic model is explored.

Keywords: Hopf bifurcation, fractional delayed epidemic model, media coverage, asymptomatic infection, fractional proportional-derivative control.

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1 Introduction

With the emergence of infectious diseases such as SARS, EBV, and COVID-19, mathematical modeling and dynamical analysis of infectious disease models have received widespread attention in recent years. It reveals the transmission mechanism of infectious diseases and provides scientific support for epidemic prevention and control as well as public health decision-making.

Since Kermack and McKendrick [9] proposed the pioneering SIR (susceptible–infected–recovered) model in 1927, researchers have continuously improved it by incorporating various realistic factors and achieving significant progress. For example, an infectious disease model with isolation was investigated in [2, 6], showing that increased isolation can prevent epidemic spread. Jiang et al. [7, 8] analyzed pulsed vaccination in delayed models with saturated incidence and studied equilibrium properties and disease dynamics. Since asymptomatic patients may exist, their impact on disease spread was considered in [17, 18]. With the rapid development of information technology, media platforms such as television, radio, and the Internet play an important role in controlling disease transmission; results in [3, 11, 20] show that received information can change behaviour and help control outbreaks. In addition, factors such as virus mutation, recurrence, and stochastic disturbances may affect transmission coefficients to varying degrees [5, 25, 26].

Early research on the transmission dynamics of infectious diseases has focused on traditional integer-order dynamical models. With the growing understanding of infectious diseases and natural environment, it has been found that fractional-order epidemic models based on Caputo fractional derivative can more accurately describe processes with "time memory" in infectious diseases, such as the impact of the incubation period, the duration of immunity, and the cumulative effect of prevention and control measures. For example, in tuberculosis modeling, researchers estimated an optimal fractional order $q = 0.93$, which reduced the model's fitting error for China's 2005–2016 actual data by 28.5% compared to integer-order models, significantly improving prediction accuracy. A fractional-order SEIQRDP model ($q = 0.96 \sim 1.02$), constructed using COVID-19 data from Canada and incorporating isolated individuals, deceased cases, and protected populations, shows a 12.7% improvement in goodness-of-fit. Therefore, kinetic studies of fractional-order epidemic models have attracted considerable attention, and numerous notable works have been reported in recent years (see [10,13,15,21,30] and the references therein). Khajji et al. [10] extended the classical SEIR model to consider the fractional-order COVID-19 model with age structure and further analyzed the order of fractional derivative on the efficiency of the control strategy. Liu et al. [15] studied the existence and stability of disease-free equilibrium and endemic equilibrium point in a fractional delayed SEIR model with an isolated term, and explored the effects of time delays and the order of fractional derivatives on disease transmission.

It is well known that Hopf bifurcation is a very important bifurcation phenomenon in nonlinear dynamical systems and can make it feasible to obtain the information of periodic solutions via Hopf bifurcation scheme. So, the existence and direction of Hopf bifurcation for various SIR models have attracted the attention of some scholars [1, 12, 14, 23, 27, 28]. Bifurcation control aims to modify the stability of the equilibrium points and adjust the bifurcation threshold to a desirable value by designing an appropriate controller. The bifurcation analysis and optimal control were carried out for a COVID-19 SIR model in [1]. Based on feedback control method, the Hopf bifurcation of delayed fractional eco-epidemiological systems was investigated in [14, 23]. So far, many kinds of control strategies have been effectively applied to nonlinear dynamical systems, such as proportional-derivative (PD) control [4], proportional-integral-derivative (PID) control [24], and hybrid control [29]. However, these methods have mainly been applied to Hopf bifurcation control in neural network systems and population ecosystems, and have seldom been used for infectious disease models up till now. This fact motivated us to study Hopf bifurcation of a novelly proposed fractional epidemic model by using fractional PD control method, which contains more adjustable parameters and can improve the speed of response of the controlled system more effectively compared with integer-order PD controller.

In this paper, we build a class of fractional epidemic model including asymptomatic patients and media coverage effect based on the proposed model in [22], and then design a fractional PD controller to optimize bifurcation behavior. The remainder of this paper is organized as follows. First, we will give a detailed description for our proposed epidemic model. Then some necessary definitions and lemmas are presented in Section 3 for later analysis. In Section 4, the stability and Hopf bifurcation problems are intensively discussed based on fractional PD control method by taking the involved time delay and the order of fractional derivatives as the bifurcation parameter, respectively. A simulation example is provided in Section 5 to illustrate our theoretical results. In Section 6, some key conclusions and the contributions of this paper are stated in brief.

2 Model formulation

More recently, Wang et al. [22] proposed a integer-order epidemic model incorporating asymptomatic patients and media coverage. According to their model, patients were divided into symptomatic patients and the asymptomatic patients. A susceptible individual was infected with a certain probability by contacting with symptomatic or asymptomatic patients. Moreover, the impact of media coverage on the dynamics of infectious disease transmission was also taken into account in their model. In view of the fact that a fractional epidemic model coincides more effectively with the genetic and memory characteristics of the infectious disease transmission and has higher precision fitting ability, we generalize this pioneering integer-order model to a fractional model, which is described as follows:

$$\begin{aligned}
D^q S(t) &= a_1 - a_2 \mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t) + I_a(t)\big]S(t) - a_3 S(t) + a_4 I_s(t) + a_5 I_a(t),\\
D^q I_s(t) &= b_1 a_2 \mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t) + I_a(t)\big]S(t) - (a_3 + a_4) I_s(t) + b_2 I_a(t),\\
D^q I_a(t) &= (1-b_1) a_2 \mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t) + I_a(t)\big]S(t) - (a_3 + a_5 + b_2) I_a(t),\\
D^q M(t) &= -c_1 M(t) + c_2\big[I_s(t-\tau_2) + I_a(t-\tau_2)\big].
\end{aligned} \tag{1}$$

Here the fractional order $q \in (0, 1]$, $S(t)$, $I_s(t)$, and $I_a(t)$ denote the number of susceptible individuals, symptomatic patients, and the asymptomatic patients at time t, respectively.

Table 1

Variable/ Parameter	Biological meaning
a_1	The input rate
a_2	Patient's daily contact rate
a_3	The natural mortality rate of the population
a_4	The cure rate of symptomatic patients
a_5	The self-healing rate of asymptomatic patients
b_1	The proportion of symptomatic patients in total patients
b_2	The transformation rate from asymptomatic patients to symptomatic patients
c_1	The dissipation rate of the media information

The media coverage is introduced as an independent variable, and $M(t)$ denotes the accumulated media information volume at time t. The changing rate of the media information volume is assumed to depend on the number of patients at time $t - \tau_2$ with a rate c_2. $\tau_1 > 0$ represents the time delay of susceptible individuals' feedback after media coverage. $\alpha > 0$ determines the degree to which media coverage influences the infectious disease transmission. The biological meanings of the other parameters in system (1) are defined in Table 1.

Remark 1. Despite the epidemic model (1) considered in this paper is only a generalization of the proposed model in [22] from an integer-order differential equation to a fractional-order differential equation, it can better describe the transmission mechanism of infectious diseases; meanwhile, it also renders the analytical technique used in [22] inapplicable to our study. Moreover, the focus of our paper is to introduce appropriate fractional PD controllers to improve the stability of the system and modify the bifurcation characteristics of the controlled system more effectively.

3 Priori knowledge

In this section, the definition of the Caputo fractional derivative and some lemmas are presented for the sake of later analysis of the stability and Hopf bifurcation of system (1).

Definition 1. (See [19].) The Caputo fractional derivative of order q of a function $f(t) \in C^m([t_0, +\infty), \mathbb{R})$ is defined as

$$D^q f(t) = \frac{1}{\Gamma(m-q)} \int_{t_0}^{t} (t-s)^{m-q-1} f^{(m)}(s)\,\mathrm{d}s,$$

where $t_0 < t$, $m - 1 < q \leqslant m$ $(m \in \mathbb{N}^+)$, $\Gamma(\cdot)$ is the gamma function. In particular, when $0 < q \leqslant 1$,

$$D^q f(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} f'(s)\,\mathrm{d}s.$$

If $f^{(l)}(0) = 0$ for $l = 0, \dots, m-1$, then the Laplace transform of $D^q f(t)$ is given by $\mathcal{L}[D^q f(t); s] = s^q F(s)$.

Lemma 1. (*See* [16].) *Consider the following autonomous fractional-order dynamical system*:

$$D^q x(t) = f\bigl(x(t)\bigr), \qquad x(0) = x_0, \tag{2}$$

where $q \in (0, 1]$, *and* $f : \mathbb{R}^m \to \mathbb{R}^m$. *Let* $J = \partial f/\partial x$ *denote the Jacobian matrix of* $f(\cdot)$ *evaluated at the equilibrium point* $x = x^*$. *If all eigenvalues* λ *of* J *satisfy* $|\arg(\lambda)| > q\pi/2$, *then the equilibrium point* $x = x^*$ *of system* (2) *is locally asymptotically stable.*

(A_1) Assume that all the parameters in system (1) satisfy the following conditions:

(i) $b_1(a_5 - a_4) + (a_3 + a_4 + b_2) > 0$;

(ii) $(a_3 + a_5 + b_2)(a_3 + a_4)/(b_1(a_5 - a_4) + (a_3 + a_4 + b_2)) < a_1 a_2/a_3$.

Lemma 2. *Under assumption* (A_1), *system* (1) *has at least one positive equilibrium point* (S^*, I_s^*, I_a^*, M^*).

Proof. The equilibrium point (S^*, I_s^*, I_a^*, M^*) of system (1) should satisfy the following equations:

$$a_1 - a_2 \mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^* - a_3 S^* + a_4 I_s^* + a_5 I_a^* = 0, \tag{3}$$

$$b_1 a_2 \mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^* - (a_3 + a_4)I_s^* + b_2 I_a^* = 0, \tag{4}$$

$$(1 - b_1)a_2 \mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^* - (a_3 + a_5 + b_2)I_a^* = 0, \tag{5}$$

$$-c_1 M^* + c_2(I_s^* + I_a^*) = 0. \tag{6}$$

Adding Eqs. (3)–(7) yields $S^* + I_s^* + I_a^* = a_1 a_3$. Combining this with Eq. (6), we obtain

$$S^* = \frac{a_1}{a_3} - \frac{c_1}{c_2}M^*. \tag{7}$$

It follows from Eqs. (4) and (7) that $I_s^* = (b_1(a_3 + a_5) + b_2)/((1 - b_1)(a_3 + a_4))I_a^*$, then we obtain

$$I_a^* = \frac{c_1(1 - b_1)(a_3 + a_4)}{c_2 b_1(a_5 - a_4) + c_2(a_3 + a_4 + b_2)}M^* \tag{8}$$

and

$$I_s^* = \frac{c_1 b_1(a_3 + a_5) + c_1 b_2}{c_2 b_1(a_5 - a_4) + c_2(a_3 + a_4 + b_2)}M^*. \tag{9}$$

By substituting $I_s^* + I_a^* = (c_1/c_2)M^*$, (8), and (9) into Eq. (7) we have

$$a_2 \mathrm{e}^{-\alpha M^*}\left(\frac{a_1}{a_3} - \frac{c_1}{c_2}M^*\right) - \frac{(a_3 + a_5 + b_2)(a_3 + a_4)}{b_1(a_5 - a_4) + (a_3 + a_4 + b_2)} = 0. \tag{10}$$

Let

$$f(x) = \frac{(a_3 + a_5 + b_2)(a_3 + a_4)}{b_1(a_5 - a_4) + (a_3 + a_4 + b_2)}\mathrm{e}^{\alpha x} + \frac{a_2 c_1}{c_2}x - \frac{a_1 a_2}{a_3}.$$

If assumption (A_1) is satisfied, we have $\lim_{x\to+\infty} f(x) = +\infty$ and $f(0) < 0$. It follows from the continuity of $f(x)$ that there exists at least one positive constant x^* such that $f(x^*) = 0$, i.e., there exists at least one positive constant M^* satisfying (10). This proves the existence of M^* and, consequently, the existence of S^*, I_s^*, and I_a^* based on (7)–(9). This completes the proof. □

To optimize the dynamical characteristics of bifurcation, we will apply the following PD^q feedback controllers

$$U_1(t) = k_{11}D^q[I_s(t) - I_s^*] + k_{12}\big[I_s(t-\tau) - I_s^*\big] \tag{11}$$

and

$$U_2(t) = k_{21}D^q[I_a(t) - I_a^*] + k_{22}\big[I_a(t-\tau) - I_a^*\big], \tag{12}$$

to the symptomatic patients and asymptomatic patients, respectively. Thus the fractional epidemic model (1) with controllers (11) and (12) can be described as follows:

$$\begin{aligned} D^qS(t) &= a_1 - a_2\mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t) + I_a(t)\big]S(t) - a_3S(t) + a_4I_s(t) + a_5I_a(t),\\ D^qI_s(t) &= \frac{1}{1-k_{11}}\big\{b_1a_2\mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t)+I_a(t)\big]S(t) - (a_3+a_4)I_s(t),\\ &\quad + b_2I_a(t) + k_{12}\big[I_s(t-\tau) - I_s^*\big]\big\},\\ D^qI_a(t) &= \frac{1}{1-k_{21}}\big\{(1-b_1)a_2\mathrm{e}^{-\alpha M(t-\tau_1)}\big[I_s(t)+I_a(t)\big]S(t)\\ &\quad - (a_3+a_5+b_2)I_a(t) + k_{22}\big[I_a(t-\tau) - I_a^*\big]\big\},\\ D^qM(t) &= -c_1M(t) + c_2\big[I_s(t-\tau_2) + I_a(t-\tau_2)\big], \end{aligned} \tag{13}$$

where it is assumed that $\tau = \tau_1 + \tau_2$, and $k_{i1}, k_{i2} < 1$ $(i = 1, 2)$ denote the derivative gain and the proportional gain, respectively.

Linearizing system (13) around the equilibrium point (S^*, I_s^*, I_a^*, M^*) and making the variable substitutions $\hat{S}(t) = S(t) - S^*$, $\hat{I}_s(t) = I_s(t) - I_s^*$, $\hat{I}_a(t) = I_a(t) - I_a^*$, and $\hat{M}(t) = M(t) - M^*$ yield

$$\begin{aligned} D^q\hat{S}(t) &= l_{11}\hat{S}(t) + l_{12}\hat{I}_s(t) + l_{13}\hat{I}_a(t) + l_{14}\hat{M}(t-\tau_1),\\ D^q\hat{I}_s(t) &= l_{21}\hat{S}(t) + l_{22}\hat{I}_s(t) + l_{23}\hat{I}_s(t-\tau) + l_{24}\hat{I}_a(t) + l_{25}\hat{M}(t-\tau_1),\\ D^q\hat{I}_a(t) &= l_{31}\hat{S}(t) + l_{32}\hat{I}_s(t) + l_{33}\hat{I}_a(t) + l_{34}\hat{I}_a(t-\tau) + l_{35}\hat{M}(t-\tau_1),\\ D^q\hat{M}(t) &= l_{41}\hat{S}(t) + l_{42}\hat{I}_s(t-\tau_2) + l_{43}\hat{I}_a(t-\tau_2) + l_{44}\hat{M}(t), \end{aligned} \tag{14}$$

where

$$\begin{gathered} l_{11} = -a_2\mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*) - a_3, \qquad l_{12} = -a_2\mathrm{e}^{-\alpha M^*}S^* + a_4,\\ l_{13} = -a_2\mathrm{e}^{-\alpha M^*}S^* + a_5, \qquad l_{14} = \alpha a_2\mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^*,\\ l_{21} = a_2b_1\mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*), \qquad l_{22} = \frac{a_2b_1\mathrm{e}^{-\alpha M^*}S^* - (a_3+a_4)}{1-k_{11}}, \end{gathered}$$

$$l_{23} = \frac{k_{12}}{1-k_{11}}, \qquad l_{24} = \frac{a_2 b_1 \mathrm{e}^{-\alpha M^*} S^* + b_2}{1-k_{11}},$$
$$l_{25} = -\frac{\alpha a_2 \mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^*}{1-k_{11}}, \qquad l_{31} = \frac{a_2(1-b_1)\mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)}{1-k_{21}},$$
$$l_{32} = \frac{a_2(1-b_1)\mathrm{e}^{-\alpha M^*} S^*}{1-k_{21}}, \qquad l_{33} = \frac{a_2(1-b_1)\mathrm{e}^{-\alpha M^*} S^* - (a_3 + a_5 + b_2)}{1-k_{21}},$$
$$l_{34} = \frac{k_{22}}{1-k_{21}}, \qquad l_{35} = -\frac{\alpha(1-b_1) a_2 \mathrm{e}^{-\alpha M^*}(I_s^* + I_a^*)S^*}{1-k_{21}},$$
$$l_{41} = 0, \qquad l_{42} = l_{43} = c_2, \qquad l_{42} = -c_1.$$

Thus, the stability of the equilibrium point (S^*, I_s^*, I_a^*, M^*) of system (13) is equivalent to the stability of the zero equilibrium point of system (14).

4 Stability and Hopf bifurcation analysis

In this section, the stability and Hopf bifurcation of the controlled system (13) will be investigated in depth. Some sufficient bifurcation conditions will be given by taking the involved time delay as the bifurcation parameter and the order of fractional derivative as the bifurcation parameter, respectively.

4.1 τ-induced Hopf bifurcation analysis

In this subsection, we will take the time delay τ (assumed to satisfy $\tau = \tau_1 + \tau_2$) as the bifurcation parameter. Some sufficient conditions for Hopf bifurcation, as well as the critical value of τ, will be given. First, the stability of system (13) needs to be discussed for the case $\tau = 0$.

It should be mentioned that if $\tau = 0$, then $\tau_1 = \tau_2 = 0$ due to the assumptions $\tau = \tau_1 + \tau_2$ and $\tau_1, \tau_2 > 0$. So, when $\tau = 0$, the characteristic equation of system (14) can be obtained as

$$\lambda^4 + \mathcal{K}_3 \lambda^3 + \mathcal{K}_2 \lambda^2 + \mathcal{K}_1 \lambda + \mathcal{K}_0 = 0, \tag{15}$$

where

$$\begin{aligned}
\mathcal{K}_0 = {} & l_{11} l_{44}(l_{22} + l_{23})(l_{33} + l_{34}) + l_{11}\big[l_{24} l_{35} l_{42} + l_{25} l_{43} l_{32} - l_{24} l_{35} l_{42} \\
& + l_{43} l_{35}(l_{22} + l_{23}) + l_{42} l_{25}(l_{33} + l_{34}) + l_{44} l_{24} l_{32}\big] + l_{21}\big[l_{14} l_{42}(l_{33} + l_{34}) \\
& + l_{44} l_{13} l_{32} + l_{34} l_{43} l_{12} - l_{33} l_{44} l_{12} - l_{13} l_{34} l_{42} - l_{14} l_{43} l_{32}\big] + l_{31}\big[l_{12} l_{24} l_{44} \\
& + l_{13} l_{25} l_{42} + l_{14} l_{43}(l_{22} + l_{23}) - l_{12} l_{25} l_{43} - l_{13} l_{44}(l_{22} + l_{23}) - l_{14} l_{42} l_{24}\big],
\end{aligned} \tag{16}$$

$$\begin{aligned}
\mathcal{K}_1 = {} & (l_{11} + l_{22} + l_{23})\big[l_{35} l_{43} - (l_{33} + l_{34}) l_{44}\big] \\
& + (l_{33} + l_{34} + l_{44})\big[l_{12} l_{21} - l_{11}(l_{22} + l_{23})\big] + l_{13} l_{31}(l_{22} + l_{23} + l_{44}) \\
& + l_{24} l_{32}(l_{11} + l_{44}) + l_{25} l_{42}(l_{11} + l_{33} + l_{34}) \\
& - (l_{12} l_{24} l_{31} + l_{13} l_{32} l_{21} + l_{14} l_{42} l_{21} + l_{24} l_{35} l_{42} + l_{25} l_{43} l_{32} + l_{14} l_{43} l_{31}),
\end{aligned} \tag{17}$$

$$\mathcal{K}_2 = (l_{11}+l_{22}+l_{23})(l_{33}+l_{34}+l_{44}) + l_{11}(l_{22}+l_{23}) + (l_{33}+l_{34})l_{44} \\ - (l_{12}l_{21}+l_{13}l_{31}+l_{24}l_{32}+l_{25}l_{42}+l_{35}l_{43}), \tag{18}$$

$$\mathcal{K}_3 = -(l_{11}+l_{22}+l_{23}+l_{33}+l_{34}+l_{44}). \tag{19}$$

According to Lemma 1, the following assumption (A_2) is needed to ensure the local asymptotical stability of the equilibrium point (S^*, I_s^*, I_a^*, M^*) of system (13) when $\tau=0$.

(A_2) Assume that $\mathcal{K}_i$ ($i=0,1,2,3$), defined earlier in (16)–(19), satisfy the following conditions:

(i) $\mathcal{K}_i > 0$ ($i=0,1,2,3$);
(ii) $\mathcal{K}_2\mathcal{K}_3 > \mathcal{K}_1$;
(iii) $\mathcal{K}_1\mathcal{K}_2\mathcal{K}_3 > \mathcal{K}_1^2 + \mathcal{K}_0\mathcal{K}_3^2$.

Remark 2. According to Routh–Hurwitz criterion, if assumption (A_2) is satisfied, then all the roots of Eq. (15) have negative real parts, i.e., one can have $|\arg(\lambda)| > \pi/2 > q\pi/2$.

To explore the stability and Hopf bifurcation of system (13), we first derive the following characteristic matrix by applying the Laplace transform to both sides of (14):

$$\Delta(s) = \begin{pmatrix} s^q - l_{11} & -l_{12} & -l_{13} & -l_{14}\mathrm{e}^{-s\tau_1} \\ -l_{21} & s^q - (l_{22}+l_{23}\mathrm{e}^{-s\tau}) & -l_{24} & -l_{25}\mathrm{e}^{-s\tau_1} \\ -l_{31} & -l_{32} & s^q - (l_{33}+l_{34}\mathrm{e}^{-s\tau}) & -l_{35}\mathrm{e}^{-s\tau_1} \\ 0 & -l_{42}\mathrm{e}^{-s\tau_2} & -l_{43}\mathrm{e}^{-s\tau_2} & s^q - l_{44} \end{pmatrix}.$$

It follows from $\det(\Delta(s)) = 0$ that

$$P_0(s,q) + P_1(s,q)\mathrm{e}^{-s\tau} + P_2(s,q)\mathrm{e}^{-2s\tau} = 0, \tag{20}$$

where

$$\begin{aligned} P_0(s,q) &= s^{4q} + \Xi_{30}s^{3q} + \Xi_{20}s^{2q} + \Xi_{10}s^q + \Xi_{00}, \\ P_1(s,q) &= \Xi_{31}s^{3q} + \Xi_{21}s^{2q} + \Xi_{11}s^q + \Xi_{01}, \\ P_2(s,q) &= \Xi_{22}s^{2q} + \Xi_{12}s^q + \Xi_{02} \end{aligned}$$

and

$$\begin{aligned} \Xi_{30} &= -(l_{11}+l_{22}+l_{33}+l_{44}), \qquad \Xi_{31} = -(l_{23}+l_{24}), \\ \Xi_{20} &= (l_{22}+l_{33})(l_{11}+l_{44}) + l_{11}l_{44} + l_{22}l_{33} - (l_{12}l_{21}+l_{13}l_{31}+l_{24}l_{32}), \\ \Xi_{21} &= (l_{23}+l_{24})(l_{11}+l_{44}) + l_{22}l_{34} + l_{23}l_{33} - l_{25}l_{42} - l_{35}l_{43}, \qquad \Xi_{22} = l_{23}l_{34}, \\ \Xi_{10} &= l_{12}(l_{21}l_{33}+l_{31}l_{24}) + l_{13}(l_{22}l_{31}-l_{21}l_{32}) + l_{44}(l_{21}l_{12}+l_{31}l_{13}) \\ &\quad + l_{24}l_{32}(l_{11}+l_{44}) - l_{11}l_{22}(l_{33}+l_{44}) - l_{33}l_{44}(l_{11}+l_{22}), \\ \Xi_{11} &= l_{21}(l_{12}l_{34}-l_{14}l_{42}) + l_{31}(l_{13}l_{23}-l_{14}l_{43}) + l_{25}l_{42}(l_{11}+l_{33}) + l_{35}l_{43}(l_{11}+l_{22}) \\ &\quad - l_{11}l_{44}(l_{23}+l_{24}) - (l_{11}+l_{44})(l_{22}l_{34}+l_{23}l_{33}) - l_{24}l_{42}l_{35} - l_{25}l_{32}l_{43}, \\ \Xi_{12} &= l_{25}l_{34}l_{42} + l_{23}l_{35}l_{43} - l_{23}l_{34}(l_{11}+l_{44}), \\ \Xi_{00} &= l_{44}\big[l_{11}(l_{22}l_{33}-l_{24}l_{32}) + l_{21}(l_{13}l_{32}-l_{12}l_{33}) + l_{31}(l_{12}l_{24}-l_{13}l_{22})\big], \end{aligned}$$

$$\begin{aligned}\varXi_{01} &= (l_{34}l_{44} - l_{35}l_{43})(l_{11}l_{22} - l_{12}l_{21}) + (l_{32}l_{43} - l_{42}l_{33})(l_{11}l_{25} - l_{14}l_{21}) \\ &\quad + l_{23}l_{44}(l_{11}l_{33} - l_{13}l_{31}) + l_{31}l_{42}(l_{13}l_{25} - l_{14}l_{24}) + l_{31}l_{43}(l_{14}l_{22} - l_{12}l_{25}) \\ &\quad + l_{35}l_{42}(l_{11}l_{24} - l_{13}l_{21}), \\ \varXi_{02} &= l_{11}(l_{23}l_{34}l_{44} - l_{23}l_{35}l_{43} - l_{25}l_{34}l_{42}) + l_{14}(l_{21}l_{34}l_{42} + l_{23}l_{31}l_{43}).\end{aligned}$$

Multiplying both sides of Eq. (20) by $\mathrm{e}^{s\tau}$ gives

$$P_0(s,q)\mathrm{e}^{s\tau} + P_1(s,q) + P_2(s,q)\mathrm{e}^{-s\tau} = 0. \tag{21}$$

Assume that $s = \mathrm{i}\omega$ ($\omega > 0$) is a root of Eq. (21) and substitute $s = (\cos\pi/2 + \mathrm{i}\sin\pi/2)\omega$ into $P_k(s,q)$, $k = 0,1,2$. Denote $P_k(\mathrm{i}\omega, q) = A_k + \mathrm{i}B_k$, $k = 0,1$. Separating the real and imaginary parts of (21), we obtain

$$\begin{aligned}(A_0 + A_2)\cos\omega\tau + (B_2 - B_0)\sin\omega\tau &= -A_1 \\ (B_0 + B_2)\cos\omega\tau + (A_0 - A_2)\sin\omega\tau &= -B_1,\end{aligned} \tag{22}$$

where

$$\begin{aligned}A_0 &= \omega^{4q}\cos 2q\pi + \varXi_{30}\omega^{3q}\cos\frac{3q\pi}{2} + \varXi_{20}\omega^{2q}\cos q\pi + \varXi_{10}\omega^{q}\cos\frac{q\pi}{2} + \varXi_{00}, \\ B_0 &= \omega^{4q}\sin 2q\pi + \varXi_{30}\omega^{3q}\sin\frac{3q\pi}{2} + \varXi_{20}\omega^{2q}\sin q\pi + \varXi_{10}\omega^{q}\sin\frac{q\pi}{2}, \\ A_1 &= \varXi_{31}\omega^{3q}\cos\frac{3q\pi}{2} + \varXi_{21}\omega^{2q}\cos q\pi + \varXi_{11}\omega^{q}\cos\frac{q\pi}{2} + \varXi_{01}, \\ B_1 &= \varXi_{31}\omega^{3q}\sin\frac{3q\pi}{2} + \varXi_{21}\omega^{2q}\sin q\pi + \varXi_{11}\omega^{q}\sin\frac{q\pi}{2}, \\ A_2 &= \varXi_{22}\omega^{2q}\cos q\pi + \varXi_{12}\omega^{q}\cos\frac{q\pi}{2} + \varXi_{02}, \\ B_2 &= \varXi_{22}\omega^{2q}\sin q\pi + \varXi_{12}\omega^{q}\sin\frac{q\pi}{2}.\end{aligned}$$

It is easy to obtain from (22) that

$$\begin{aligned}\cos\omega\tau &= \frac{A_1(A_2 - A_0) + B_1(B_2 - B_0)}{A_0^2 + B_0^2 - A_2^2 - B_2^2} = H_1(\omega), \\ \sin\omega\tau &= \frac{A_1(B_2 + B_0) - B_1(A_2 + A_0)}{A_0^2 + B_0^2 - A_2^2 - B_2^2} = H_2(\omega),\end{aligned}$$

Using the trigonometric identity $\sin^2\omega\tau + \cos^2\omega\tau = 1$, one can obtain the following ω-dependent equation:

$$H_1^2(\omega) + H_2^2(\omega) = 1. \tag{23}$$

Assume that Eq. (23) has at least one positive root ω, which can be obtained using the numerical software Maple 13. Then the critical value of bifurcation point τ_0 is defined as

$$\tau_0 = \frac{1}{\omega}\min_{k=0,1,2,\dots}\bigl(\arccos H_1(\omega) + 2k\pi\bigr), \tag{24}$$

where ω is the root of Eq. (23).

In what follows, the bifurcation of system (13) induced by τ will be further analyzed. To this aim, we need the following assumption (A_3).

(A_3) $(\mathit{\Gamma}_1\mathit{\Upsilon}_1+\mathit{\Gamma}_2\mathit{\Upsilon}_2)/\mathit{\Upsilon}_1^2+\mathit{\Upsilon}_2^2 \neq 0$, where $\mathit{\Gamma}_k, \mathit{\Upsilon}_k$ $(k=1,2)$ are defined by (26)–(29) below, and τ_0 is defined by (24).

Lemma 3. *Let $s(\tau)=\gamma(\tau)+\mathrm{i}\omega(\tau)$ be a root of Eq.* (20) *near $\tau=\tau_0$, satisfying $\gamma(\tau_0)=0$ and $\omega(\tau_0)=\omega_0>0$. Then, under assumption* (A_3)*, the following transversality condition is satisfied*:

$$\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\bigg|_{\tau=\tau_0,\,\omega=\omega_0} \neq 0.$$

Proof. Calculating the derivative of s with respect to τ on both sides of (20), we have

$$P_0'(s,q)\frac{\mathrm{d}s}{\mathrm{d}\tau} - P_1(s,q)\mathrm{e}^{-s\tau}\left(s+\tau\frac{\mathrm{d}s}{\mathrm{d}\tau}\right) + P_1'(s,q)\mathrm{e}^{-s\tau}\frac{\mathrm{d}s}{\mathrm{d}\tau}$$
$$-2P_2(s,q)\mathrm{e}^{-2s\tau}\left(s+\tau\frac{\mathrm{d}s}{\mathrm{d}\tau}\right) + P_2'(s,q)\mathrm{e}^{-2s\tau}\frac{\mathrm{d}s}{\mathrm{d}\tau} = 0.$$

Thus it can be obtained that

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{s[P_1(s,q)\mathrm{e}^{-s\tau}+2P_2(s,q)\mathrm{e}^{-2s\tau}]}{P_0'(s,q)+P_1'(s,q)\mathrm{e}^{-s\tau}+P_2'(s,q)\mathrm{e}^{-2s\tau}-\tau[P_1(s,q)\mathrm{e}^{-s\tau}+2P_2(s,q)\mathrm{e}^{-2s\tau}]}. \tag{25}$$

Denote $P_k(\mathrm{i}\omega_0,q)=\mu_k+\mathrm{i}\varepsilon_k$ $(k=1,2)$, $P_l'(\mathrm{i}\omega_0,q)| = \hbar_l+\mathrm{i}\ell_l$ $(l=0,1,2)$. Substituting them into (25) yields

$$\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\bigg|_{\tau=\tau_0,\,\omega=\omega_0} = \frac{\mathit{\Gamma}_1\mathit{\Upsilon}_1+\mathit{\Gamma}_2\mathit{\Upsilon}_2}{\mathit{\Upsilon}_1^2+\mathit{\Upsilon}_2^2},$$

where

$$\mathit{\Gamma}_1 = -\omega_0\big[\varepsilon_1\cos\omega_0\tau_0 - \mu_1\sin\omega_0\tau_0 + 2(\varepsilon_2\cos 2\omega_0\tau_0 - \mu_2\sin 2\omega_0\tau_0)\big], \tag{26}$$

$$\mathit{\Gamma}_2 = \omega_0\big[\mu_1\cos\omega_0\tau_0 + \varepsilon_1\sin\omega_0\tau_0 + 2(\mu_2\cos 2\omega_0\tau_0 + \varepsilon_2\sin 2\omega_0\tau_0)\big], \tag{27}$$

$$\begin{aligned}\mathit{\Upsilon}_1 &= \hbar_0 + (\hbar_1-\mu_1\tau_0)\cos\omega_0\tau_0 + (\ell_1-\varepsilon_1\tau_0)\sin\omega_0\tau_0 \\ &\quad + (\hbar_2-2\mu_2\tau_0)\cos 2\omega_0\tau_0 + (\ell_2-2\varepsilon_2\tau_0)\sin 2\omega_0\tau_0,\end{aligned} \tag{28}$$

$$\begin{aligned}\mathit{\Upsilon}_2 &= \ell_0 + (\ell_1-\varepsilon_1\tau_0)\cos\omega_0\tau_0 - (\hbar_1-\mu_1\tau_0)\sin\omega_0\tau_0 \\ &\quad + (\ell_2-2\varepsilon_2\tau_0)\cos 2\omega_0\tau_0 - (\hbar_2-2\mu_2\tau_0)\sin 2\omega_0\tau_0.\end{aligned} \tag{29}$$

If assumption (A_3) is satisfied, $\mathrm{Re}(\mathrm{d}s/\mathrm{d}\tau)|_{\tau=\tau_0,\,\omega=\omega_0} \neq 0$. This completes the proof. □

Combining above discussion with Hopf bifurcation theorem, we can get the following Theorem 1.

Theorem 1. *Suppose that assumptions* (A_2) *and* (A_3) *are satisfied. Then the equilibrium point (S^*, I_s^*, I_a^*, M^*) of the controlled system* (13) *is asymptotically stable when $\tau \in [0,\tau_0)$, and a Hopf bifurcation occurs near this equilibrium point when $\tau=\tau_0$. Here τ_0 is defined in* (24).

4.2 q-induced Hopf bifurcation analysis

In this section, we will take the order q as the bifurcation parameter. Some sufficient Hopf bifurcation criteria, as well as the critical value of q, will be given. Considering $q = 0$ is meaningless for a fractional dynamical system, so, in this case, we should first investigate the stability of system (13) when $q \to 0^+$.

To ensure the local stability of the controlled system (13) for the case $q \to 0^+$, we need the following assumption.

(A_4) Assume the following conditions are satisfied:

(i) $\vartheta_1 \neq \vartheta_3$;

(ii) $(\aleph_1 + \aleph_3 - \aleph_2)(2\vartheta_2 - \vartheta_1)\ln(\pi/\tau) > 0$.

Here

$$\begin{aligned}
\vartheta_1 &= 1 + \varXi_{30} + \varXi_{20} + \varXi_{10}, & \aleph_1 &= 4 + 3\varXi_{30} + 2\varXi_{20} + \varXi_{10},\\
\vartheta_2 &= \varXi_{31} + \varXi_{21} + \varXi_{11} + \varXi_{01}, & \aleph_2 &= 3\varXi_{31} + 2\varXi_{21} + \varXi_{11},\\
\vartheta_3 &= \varXi_{22} + \varXi_{12} + \varXi_{02}, & \aleph_3 &= 2\varXi_{22} + \varXi_{12},
\end{aligned}$$

and $\varXi_{ij}$ ($i = 1, 2, 3$; $j = 0, 1, 2$) are defined above.

Remark 3. It should be mentioned that condition (A_4)(ii) includes two cases:

$$\begin{aligned}
&(\aleph_1 + \aleph_3 - \aleph_2)(2\vartheta_2 - \vartheta_1) > 0 \quad \text{if } 0 < \tau < \pi,\\
&(\aleph_1 + \aleph_3 - \aleph_2)(2\vartheta_2 - \vartheta_1) < 0 \quad \text{if } \tau > \pi.
\end{aligned}$$

Lemma 4. *Let $s(q) = r(q) + \mathrm{i}\omega(q)$ be the root of Eq.* (20) *satisfying $\lim_{q\to 0^+} r(q) = 0$ and $\lim_{q\to 0^+} \omega(q) = \omega^* > 0$. Then, under assumption* (A_4)*, the equilibrium point (S^*, I_s^*, I_a^*, M^*) of the controlled system* (13) *is locally asymptotically stable.*

Proof. Differentiating the left-hand side of (20) with respect to q and using $\mathrm{d}s^{kq}/\mathrm{d}q = ks^{kq-1}(s\ln s + q\mathrm{d}s/\mathrm{d}q)$, $k \in \mathbb{N}$, we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}q} = -\frac{\varOmega_1(s, q)}{\varOmega_2(s, q)},$$

where $\varOmega_k(s, q)$ satisfy $\varOmega_k(\mathrm{i}\omega, q) = \varOmega_k^R(\omega, q) + \mathrm{i}\varOmega_k^I(\omega, q)$, $k = 1, 2$, and

$$\begin{aligned}
\varOmega_1^R(\omega, q) &= 4\omega^{4q}\left(\ln\omega\cos 2q\pi - \frac{\pi}{2}\sin 2q\pi\right)\\
&\quad + 3\omega^{3q}\left\{\varXi_{30}\left(\ln\omega\cos\frac{3q\pi}{2} - \frac{\pi}{2}\sin\frac{3q\pi}{2}\right)\right.\\
&\quad \left. + \varXi_{31}\left[\ln\omega\cos\left(\omega\tau - \frac{3q\pi}{2}\right) + \frac{\pi}{2}\sin\left(\omega\tau - \frac{3q\pi}{2}\right)\right]\right\}\\
&\quad + \sum_{k=1}^{2} k\omega^{kq}\left\{\varXi_{k0}\left(\ln\omega\cos\frac{kq\pi}{2} - \frac{\pi}{2}\sin\frac{kq\pi}{2}\right)\right.
\end{aligned}$$

$$+ \varXi_{k1}\left[\ln\omega\cos\left(\omega\tau - \frac{kq\pi}{2}\right) + \frac{\pi}{2}\sin\left(\omega\tau - \frac{kq\pi}{2}\right)\right]$$
$$+ \varXi_{k2}\left[\ln\omega\cos\left(2\omega\tau - \frac{kq\pi}{2}\right) + \frac{\pi}{2}\sin\left(2\omega\tau - \frac{kq\pi}{2}\right)\right]\Bigg\}, \tag{30}$$

$$\begin{aligned}
\varOmega_1^I(\omega,q) &= 4\omega^{4q}\left(\frac{\pi}{2}\cos 2q\pi + \ln\omega\sin 2q\pi\right)\\
&+ 3\omega^{3q}\Bigg\{\varXi_{30}\left(\frac{\pi}{2}\cos\frac{3q\pi}{2} + \ln\omega\sin\frac{3q\pi}{2}\right)\\
&+ \varXi_{31}\left[\frac{\pi}{2}\cos\left(\omega\tau - \frac{3q\pi}{2}\right) - \ln\omega\sin\left(\omega\tau - \frac{3q\pi}{2}\right)\right]\Bigg\}\\
&+ \sum_{k=1}^{2} k\omega^{kq}\Bigg\{\varXi_{k0}\left(\frac{\pi}{2}\cos\frac{kq\pi}{2} + \ln\omega\sin\frac{kq\pi}{2}\right)\\
&+ \varXi_{k1}\left[\frac{\pi}{2}\cos\left(\omega\tau - \frac{kq\pi}{2}\right) - \ln\omega\sin\left(\omega\tau - \frac{kq\pi}{2}\right)\right]\\
&+ \varXi_{k2}\left[\frac{\pi}{2}\cos\left(2\omega\tau - \frac{kq\pi}{2}\right) - \ln\omega\sin\left(2\omega\tau - \frac{kq\pi}{2}\right)\right]\Bigg\},
\end{aligned} \tag{31}$$

$$\begin{aligned}
\varOmega_2^R(\omega,q) &= 4q\omega^{4q-1}\cos\frac{(4q-1)\pi}{2} + 3q\varXi_{30}\omega^{3q-1}\cos\frac{(3q-1)\pi}{2}\\
&+ \sum_{k=1}^{2} kq\omega^{kq-1}\Bigg\{\varXi_{k0}\cos\frac{(kq-1)\pi}{2} + \varXi_{k1}\cos\left(\omega\tau - \frac{(kq-1)\pi}{2}\right)\\
&+ \varXi_{k2}\cos\left(2\omega\tau - \frac{(kq-1)\pi}{2}\right)\Bigg\}\\
&- \tau\Bigg\{\varXi_{31}\omega^{3q}\cos\left(\omega\tau - \frac{3q\pi}{2}\right) + (\varXi_{21} + 2\varXi_{22})\omega^{2q}\cos(\omega\tau - q\pi)\\
&+ (\varXi_{11} + 2\varXi_{12})\omega^{q}\cos\left(\omega\tau - \frac{q\pi}{2}\right) + (\varXi_{01} + 2\varXi_{02})\omega^{q}\cos\omega\tau\Bigg\},
\end{aligned} \tag{32}$$

$$\begin{aligned}
\varOmega_2^I(\omega,q) &= 4q\omega^{4q-1}\sin\frac{(4q-1)\pi}{2} + 3q\varXi_{30}\omega^{3q-1}\sin\frac{(3q-1)\pi}{2}\\
&+ \sum_{k=1}^{2} kq\omega^{kq-1}\Bigg\{\varXi_{k0}\sin\frac{(kq-1)\pi}{2} - \varXi_{k1}\sin\left(\omega\tau - \frac{(kq-1)\pi}{2}\right)\\
&- \varXi_{k2}\sin\left(2\omega\tau - \frac{(kq-1)\pi}{2}\right)\Bigg\}\\
&+ \tau\Bigg\{\varXi_{31}\omega^{3q}\sin\left(\omega\tau - \frac{3q\pi}{2}\right) + (\varXi_{21} + 2\varXi_{22})\omega^{2q}\sin(\omega\tau - q\pi)\\
&+ (\varXi_{11} + 2\varXi_{12})\omega^{q}\sin\left(\omega\tau - \frac{q\pi}{2}\right) + (\varXi_{01} + 2\varXi_{02})\omega^{q}\sin\omega\tau\Bigg\}.
\end{aligned} \tag{33}$$

Let

$$\hat{\Omega}_k^R(\omega^*) = \lim_{q\to 0^+} \Omega_k^R(\omega, q), \qquad \hat{\Omega}_k^I(\omega^*) = \lim_{q\to 0^+} \Omega_k^I(\omega, q)$$

for $k = 1, 2$. It is not difficult to obtain

$$\begin{aligned}
\hat{\Omega}_1^R(\omega^*) &= (\aleph_1 + \aleph_2 \cos\omega^*\tau + \aleph_3 \cos 2\omega^*\tau)\ln\omega^* + \frac{\pi}{2}(\aleph_2 \sin\omega^*\tau + \aleph_3 \sin 2\omega^*\tau),\\
\hat{\Omega}_1^I(\omega^*) &= \frac{\pi}{2}(\aleph_1 + \aleph_2 \cos\omega^*\tau + \aleph_3 \cos 2\omega^*\tau) - (\aleph_2 \sin\omega^*\tau + \aleph_3 \sin 2\omega^*\tau)\ln\omega^*,\\
\hat{\Omega}_2^R(\omega^*) &= \tau(\vartheta_1 \cos\omega^*\tau + 2\vartheta_2 \cos 2\omega^*\tau),\\
\hat{\Omega}_2^I(\omega^*) &= -\tau(\vartheta_1 \sin\omega^*\tau + 2\vartheta_2 \sin 2\omega^*\tau).
\end{aligned}$$

Denote $\lim_{q\to 0^+} P_k(i\omega, q) = \hat{A}_k + i\hat{B}_k$, $k = 0, 1, 2$. Then it follows that $\hat{A}_0 = \vartheta_1$, $\hat{A}_1 = \vartheta_2$, $\hat{A}_2 = \vartheta_3$, and $\hat{B}_k = 0$, $k = 0, 1, 2$. When $q \to 0^+$, Eqs. (22) degenerate into

$$(\vartheta_1 + \vartheta_3)\cos\omega^*\tau = -\vartheta_2, \qquad (\vartheta_1 - \vartheta_3)\sin\omega^*\tau = 0. \tag{34}$$

From the second equation in (34) it can be easily obtained that $\sin\omega^*\tau = 0$ or $\vartheta_1 = \vartheta_3$. According to condition (A_4)(i), we know that $\sin\omega^*\tau = 0$, which yields $\cos\omega^*\tau = 1$ or $\cos\omega^*\tau = -1$. If $\cos\omega^*\tau = 1$, then the minimum $\omega^* = 0$, which contradicts with the fact $\omega^* > 0$. Thus, we have $\cos\omega^*\tau = -1$ and, accordingly, the minimum $\omega^* = \pi/\tau$. As a result, it can be obtained from assumption (A_4)(ii) that

$$\begin{aligned}
\operatorname{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}q}\right)\bigg|_{q\to 0^+,\, \omega^*=\pi/\tau} &= -\frac{\hat{\Omega}_1^R(\omega^*)\hat{\Omega}_2^R(\omega^*) + \hat{\Omega}_1^I(\omega^*)\hat{\Omega}_2^I(\omega^*)}{(\hat{\Omega}_2^R(\omega^*))^2 + (\hat{\Omega}_2^I(\omega^*))^2}\\
&= -\frac{(\aleph_1 + \aleph_3 - \aleph_2)\ln\frac{\pi}{\tau}}{(2\vartheta_2 - \vartheta_1)\tau} < 0,
\end{aligned}$$

which indicates that all the roots of the characteristic equation (20) have negative real parts near $q \to 0^+$, and thus can ensure the local asymptotical stability of system (13) based on Lemma 1. This completes the proof. $\square$

In what follows, we will further explore the critical value of the bifurcation parameter q based on the solution of implicit function equations and Hopf bifurcation theory. To this aim, rewrite Eqs. (22) as

$$\begin{aligned}
(A_0 + A_2)\cos\omega\tau + (B_2 - B_0)\sin\omega\tau + A_1 &= 0,\\
(B_0 + B_2)\cos\omega\tau + (A_0 - A_2)\sin\omega\tau + B_1 &= 0,
\end{aligned}$$

and define

$$\Delta_1(q, \omega) = (A_0 + A_2)\cos\omega\tau + (B_2 - B_0)\sin\omega\tau + A_1, \tag{35}$$

$$\Delta_2(q, \omega) = (B_0 + B_2)\cos\omega\tau + (A_0 - A_2)\sin\omega\tau + B_1. \tag{36}$$

Denote by $\mathcal{C}_1$ and $\mathcal{C}_2$ the curves corresponding to the implicit functions $\Delta_1(q, \omega) = 0$ and $\Delta_2(q, \omega) = 0$, respectively. To make our study meaningful, here we assume that there is at least one intersection between $\mathcal{C}_1$ and $\mathcal{C}_2$ (the explicit values of all the intersections can be determined using the numerical software Maple or Matlab). The following assumption (A_5) is needed to obtain the order-induced bifurcation condition.

$$(\mathrm{A}_5) \qquad \frac{\Omega_1^R(\omega,q)\Omega_2^R(\omega,q)+\Omega_1^I(\omega,q)\Omega_2^I(\omega,q)}{(\Omega_2^R(\omega,q))^2+(\Omega_2^I(\omega,q))^2}\bigg|_{(\tilde{\omega}_0,q_0)} \neq 0,$$

where $\Omega_k^R(\omega,q)$, $\Omega_k^I(\omega,q)$, $k=1,2$, are defined in (30)–(33), $q_0=\min\{q_k\}$, $\tilde{\omega}_0$ is the value of ω corresponding to q_0, and (q_k,ω_k) are the solutions to the implicit function array composed of $\Delta_1(q,\omega)=0$ and $\Delta_2(q,\omega)=0$ ($\Delta_1(q,\omega)$ and $\Delta_2(q,\omega)$ are defined by (35) and (36)).

Theorem 2. *Suppose that assumptions* (A_1), (A_4), *and* (A_5) *are satisfied. Then the equilibrium point* (S^*,I_s^*,I_a^*,M^*) *of the controlled system* (13) *is asymptotically stable when* $q\in(0,q_0)$, *and a Hopf bifurcation occurs near this equilibrium point when* $q=q_0$.

5 Numerical results

In this section, the stability and Hopf bifurcation of a numerical example will be investigated based on the criteria proposed in Theorems 1 and 2. Meanwhile, all the simulation plots and phase portraits will be given simultaneously to show the effectiveness of the presented theoretical results.

Consider the following fractional epidemic model with PD^q feedback controllers:

$$\begin{aligned}
D^qS(t) &= 1.5-0.03\mathrm{e}^{-0.3M(t-\tau_1)}\big[I_s(t)+I_a(t)\big]S(t)-0.05S(t)\\
&\quad +0.1I_s(t)+0.2I_a(t),\\
D^qI_s(t) &= 0.018\mathrm{e}^{-0.3M(t-\tau_1)}\big[I_s(t)+I_a(t)\big]S(t)-0.15I_s(t)\\
&\quad +0.3I_a(t)+U_1(t),\\
D^qI_a(t) &= 0.012\mathrm{e}^{-0.3M(t-\tau_1)}\big[I_s(t)+I_a(t)\big]S(t)-0.55I_a(t)+U_2(t),\\
D^qM(t) &= -0.1M(t)+0.15\big[I_s(t-\tau_2)+I_a(t-\tau_2)\big],
\end{aligned} \tag{37}$$

where all the parameters satisfy assumption (A_1). Thus we can obtain the positive equilibrium point $(S^*,I_s^*,I_a^*,M^*)=(26.4647,3.1193,0.4159,5.3028)$. The PD^q feedback controllers $U_1(t)$ and $U_2(t)$ are designed as

$$\begin{aligned}
U_1(t) &= 0.08D^q\big[I_s(t)-3.1193\big]+0.1\big[I_s(t-\tau)-3.1193\big],\\
U_2(t) &= 0.11D^q\big[I_a(t)-0.4159\big]+0.2\big[I_a(t-\tau)-0.4159\big],
\end{aligned}$$

respectively. The initial value is $(S(0),I_s(0),I_a(0),M(0))=(25,3,0.4,5)$. The fractional order q and the time delays τ_1, τ_2, and τ are to be determined.

First, we concern the case of delay τ-induced Hopf bifurcation for the controlled system (37) based on the proposed Theorem 1. To this aim, select the fractional order $q=0.9$. According to (23) and (24), the critical frequency and bifurcation point can be determined as $\omega_0=0.1809$ and $\tau_0=2.6187$ with the help of numerical software Maple 13. Through some computation, we can verify that assumptions (A_2) and (A_3) are both satisfied. So, on the basis of the Theorem 1, the equilibrium point $(26.4647,3.1193,0.4159,5.3028)$ is locally asymptotically stable when $\tau\in[0,2.6187)$, and a Hopf bifurcation emerges

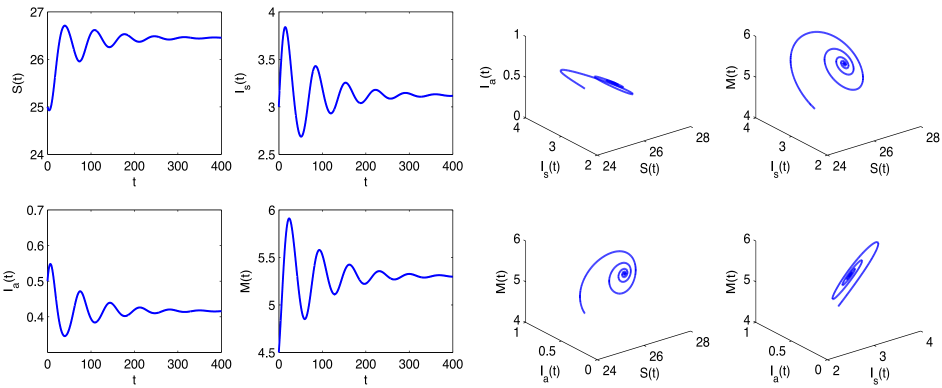

Figure 1. The simulation diagrams of system (37) when $q = 0.9$, $\tau_1 = \tau_2 = 0.8$, $\tau = 1.6 < \tau_0 = 2.6187$.

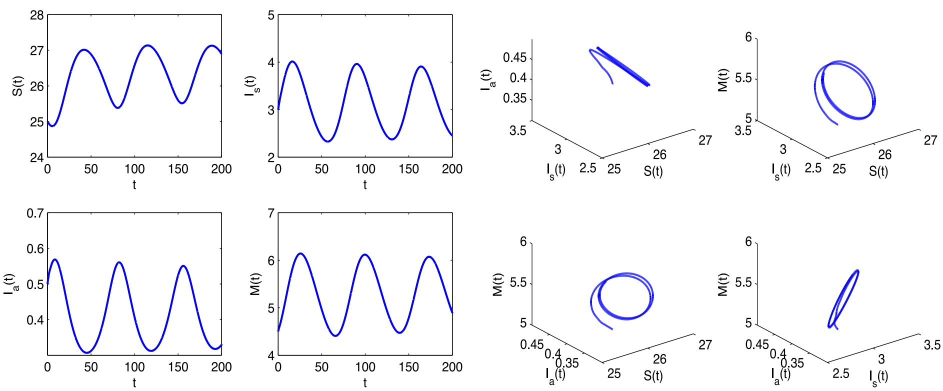

Figure 2. The simulation diagrams of system (37) when $q = 0.9$, $\tau_1 = \tau_2 = 1.5$, $\tau = 3 > \tau_0 = 2.6187$.

when $\tau \geqslant 2.6187$. The simulation plots and phase portraits of system (37) are depicted in Figs. 1 and 2 for $\tau = 1.6$ and $\tau = 3$, respectively. It can be seen from Fig. 2 that Hopf bifurcation occurs when $\tau = 3 > \tau_0$.

Then we further investigate the q-induced Hopf bifurcation for the controlled system (37) based on the given Theorem 2. In this case, we select $\tau = 4$, which ensures that assumption (A_4) is satisfied. By solving the solutions to the implicit function array composed of $\varDelta_1(q, \omega) = 0$ and $\varDelta_2(q, \omega) = 0$ (which are defined in (35) and (36)), we obtain $\tilde{\omega}_0 = 0.07671$ and $q_0 = 0.87064$, and the intersection graph of the corresponding curves $\mathcal{C}_1$ and $\mathcal{C}_2$ is showed in Fig. 3. It can be verified that assumption (A_5) is satisfied by some computation, then according to Theorem 2, the equilibrium point $(26.4647, 3.1193, 0.4159, 5.3028)$ is locally asymptotically stable when $q < 0.87064$ and loses its stability when $q > 0.87064$, a Hopf bifurcation emerges at the equilibrium point when q passes through the critical value q_0. The simulation plots and phase portraits of system (37) are depicted in Figs. 4 and 5 for $q = 0.82$ and $q = 0.92$ with $\tau_1 = \tau_2 = 2$, $\tau = 4$, respectively. It can be seen from Fig. 5 that Hopf bifurcation occurs when $q = 0.92 > q_0$.

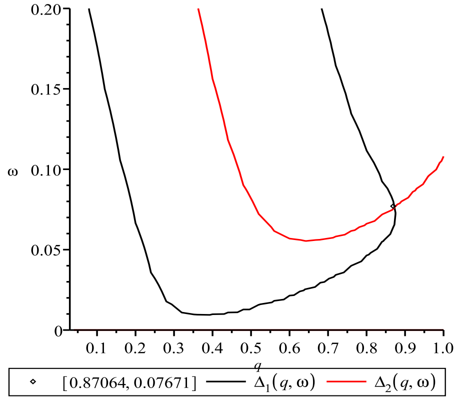

Figure 3. The intersection graph of the two curves $\mathcal{C}_1$ and $\mathcal{C}_2$ for system (37).

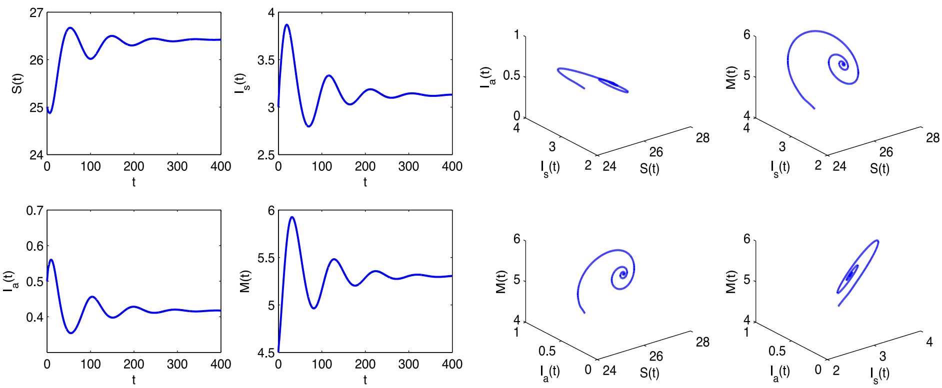

Figure 4. The simulation diagrams of system (37) when $\tau_1 = \tau_2 = 2$, $\tau = 4$, $q = 0.82 < q_0 = 0.87064$.

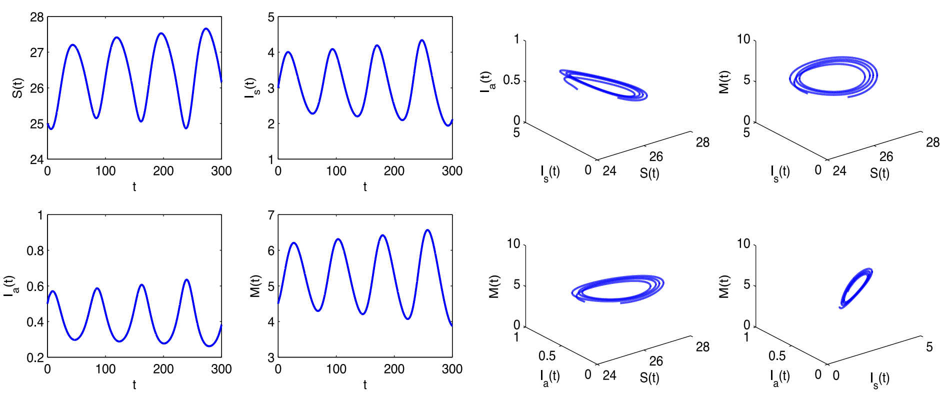

Figure 5. The simulation diageams of system (37) when $\tau_1 = \tau_2 = 2$, $\tau = 4$, $q = 0.92 > q_0 = 0.87064$.

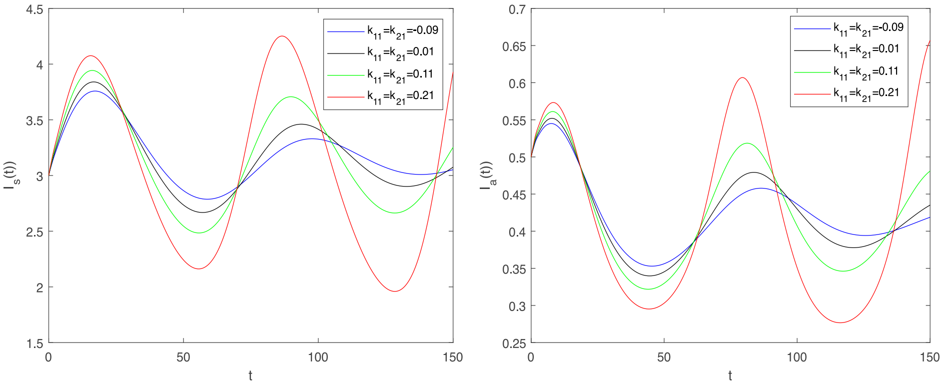

Figure 6. The waveform plots of $I_s(t)$ and $I_a(t)$ for system (37) with selected parameters except k_{11}, k_{21} when $\tau_1 = \tau_2 = 1.3$, $\tau = 2.6$, $q = 0.9$.

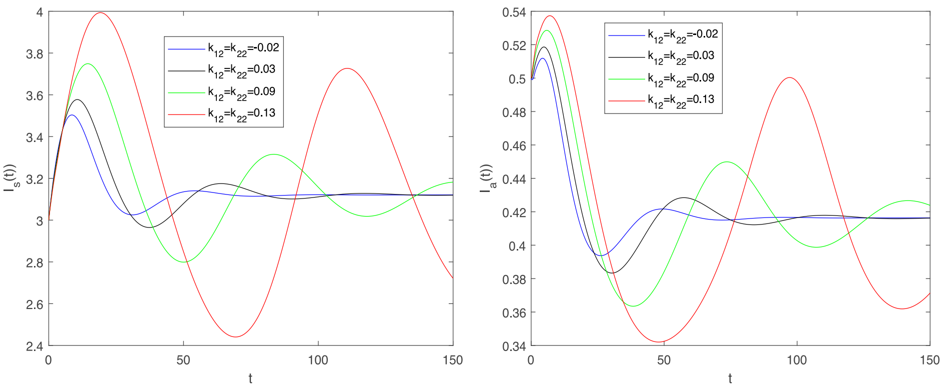

Figure 7. The waveform plots of $I_s(t)$ and $I_a(t)$ for system (37) with selected parameters except k_{12}, k_{22} when $\tau_1 = \tau_2 = 1.3$, $\tau = 2.6$, $q = 0.9$.

Remark 4. To explore the effect of the control gains on the stability of the number of patients, we consider system (37). First, we fix the values of proportional gains (derivative gains) at $k_{12} = 0.1$, $k_{22} = 0.2$ ($k_{11} = 0.08$, $k_{21} = 0.11$). Then we vary the derivative gains (proportional gains) as follows: $k_{11} = k_{21} = -0.09$, $k_{11} = k_{21} = 0.01$, $k_{11} = k_{21} = 0.11$, $k_{11} = k_{21} = 0.21$ ($k_{12} = k_{22} = -0.02$, $k_{12} = k_{22} = 0.03$, $k_{12} = k_{22} = 0.09$, $k_{12} = k_{22} = 0.13$) in turn. From Figs. 6 and 7 we can see that smaller derivative gains or proportional gains will stabilize the number of symptomatic and asymptomatic patients to the equilibrium point earlier. It can also be understood that there may be a threshold of the control gains less than which the controlled system is stable and otherwise loses stability.

Remark 5. Figure 8 shows the impact of the parameter α on the spread of the infectious disease. It can be seen that as α increases from 0.09 to 0.24, the numbers of symptomatic and asymptomatic patients at the steady state both decrease. This is because α is used to

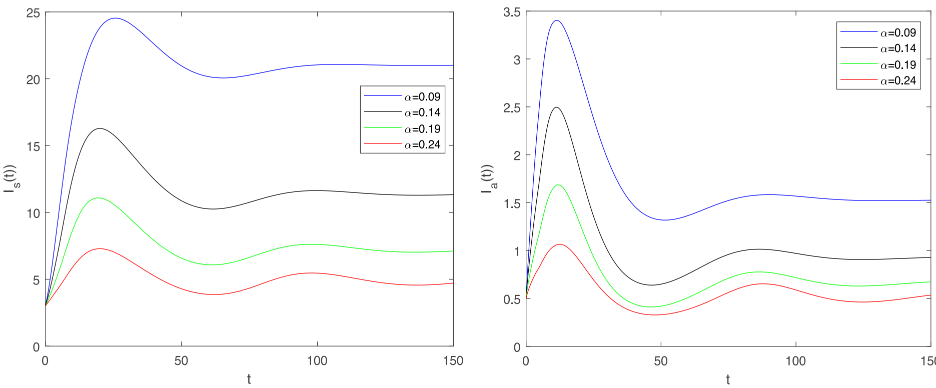

Figure 8. The waveform plots of $I_s(t)$ and $I_a(t)$ for system (37) with selected parameters except α when $\tau_1 = 3$, $\tau_2 = 1.5$, $q = 0.9$.

measure the effect of media coverage on individual behaviour change. Increasing α helps to reduce interpersonal contact and then reduce the possibility of disease transmission.

6 Conclusion

This paper focuses on the stability and Hopf bifurcation of a fractional delayed epidemic model with media coverage and asymptomatic infection. Based on fractional proportional-derivative (PD) control method, sufficient bifurcation conditions are established by taking the sum of the two time delays and the order of fractional derivatives as bifurcation parameters, respectively. The effectiveness of the proposed theoretical results is illustrated through a simulation example.

The main contribution of our work consists of three aspects:

(i) We extend the integer-order epidemic model in [22] to a fractional-order model, which enables greater accuracy and flexibility in fitting real epidemic data and predicting transmission trends. In addition, the introduced fractional-order PD^q controller provides more flexible adjustment capabilities, allowing for a more precise match to the dynamic characteristics of the controlled system.

(ii) For the considered fractional epidemic model, we first establish the existence of a positive equilibrium point, and then investigate its stability and Hopf bifurcation by taking the time delay and the order of the fractional derivatives as bifurcation parameters, respectively. Several novel sufficient bifurcation conditions are derived based on the Hopf bifurcation theory for fractional delayed dynamical systems, and the critical values of the bifurcation points are obtained using Maple 13.

(iii) We examine the effects of the feedback control gains and media coverage on stability and Hopf bifurcation through illustrative examples. The results show that smaller control gains lead to earlier stabilization of the equilibrium, and that there exists a threshold value below which the controlled system remains stable and above which it loses stability. Furthermore, the steady-state numbers of both symptomatic and asymptomatic individuals decrease as the parameter α increases, highlighting the significant impact of media coverage on the spread of the infectious disease.

Author contributions. All authors (Y.W., J.C., M.X., and L.Z) have contributed as follows: methodology, Y.W., J.C., and M.X.; formal analysis, Y.W.; software, Y.W. and L.Z.; validation, J.C. and M.X.; writing – original draft preparation, Y.W.; writing – review & editing, Y.W., J.C., and M.X. All authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

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