Nonlinear Analysis: Modelling and Control, Vol. 31, No. 4, 901–919
https://doi.org/10.15388/namc.2026.31.47341

Second-order coupled integral boundary value problems at resonance*

Yue Du, Yujun Cui[1]

College of Mathematics and Systems Science,
Shandong University of Science and Technology,
Qingdao 266590, China
cyj720201@163.com

Received: September 23, 2025 / **Revised:** March 31, 2026 / **Published online:** June 11, 2026

Abstract. This paper investigates the existence of positive solutions for a resonant system of nonlinear differential equations subject to coupled integral boundary conditions involving Riemann–Stieltjes integral. Our analysis is based on Leggett–Williams norm-type theorem for coincidence equations due to O'Regan and Zima. By employing a general abstract framework, we obtain new existence criteria that complement and extend recent results in the literature.

Keywords: positive solution, coupled integral boundary value problem, Riemann–Stieltjes integral, resonance.

1 Introduction

Coupled boundary conditions emerge naturally in the study of reaction-diffusion equations and Sturm–Liouville problems [2, 3, 15, 25], finding extensive applications across diverse scientific and engineering fields, such as in the analysis of the heat equation [9, 16] and within mathematical biology [4, 15]. Coupled boundary conditions for the ordinary differential systems was first studied by Asif and Khan [5]. Following this, a wealth of research has emerged concerning the existence, uniqueness, and solvability of coupled boundary conditions for the ordinary differential systems and fractional differential systems (see [1, 7, 12, 14, 23, 24]). For example, Asif and Khan [5] studied the existence of positive solutions to a nonlinear singular system with four-point coupled boundary conditions by the well-known Guo–Krasnosel'skii theorem on cone compression-expansion. Yuan et al. [24] investigated the existence of multiple positive solutions to systems of nonlinear semipositone fractional differential equations with coupled boundary conditions using a nonlinear alternative of Leray–Schauder type and Krasnosel'skii's fixed-point theorems. In [23], Su and Zhang obtained the existence of the positive solutions of second-order

*This research was supported by the National Natural Science Foundation of China (Nos. 12371173, 11801322) and the Shandong Provincial Natural Science Foundation (No. ZR2018MA011).

[1]Corresponding author.

coupled differential system with coupled integral boundary value conditions and nonlinearities depending on the first derivatives through posing some inequality conditions and the spectral radius conditions on the nonlinearities.

In the aforementioned papers, the addressed boundary value problems for differential systems are exclusively nonresonant. In the case of resonance, Cui [6] studied the existence of solutions for the coupled integral boundary value problem at resonance using the coincidence degree theory; Sun and Bai [21] proved the existence of solutions for the system of fractional three-point boundary value problems at resonance by employing the Moore–Penrose generalized inverse matrix. However, in [6, 21], the existence of positive solutions is not considered.

While the existence of positive solutions has been extensively studied for differential systems with coupled boundary conditions at nonresonance (see, for example, [7, 12, 14, 23, 24]) and for boundary value problems of differential equations at resonance (see, for example, [10, 13, 18, 22, 26]), the conditions for such solutions in coupled systems at resonance remain an open question. To the best of our knowledge, this problem has received little attention in the literature. The aim of this paper is to investigate the existence of positive solutions for the following coupled integral boundary value problem:

$$\begin{aligned} -\varphi''(t) &= F_1\bigl(t, \varphi(t), \psi(t)\bigr), \quad t \in (0,1), \\ -\psi''(t) &= F_2\bigl(t, \varphi(t), \psi(t)\bigr), \quad t \in (0,1), \\ \varphi'(0) &= 0, \qquad \gamma_{11}\varphi(1) + \gamma_{12}\varphi'(1) = \alpha[\psi], \\ \psi'(0) &= 0, \qquad \gamma_{21}\psi(1) + \gamma_{22}\psi'(1) = \beta[\varphi], \end{aligned} \tag{1}$$

where $\gamma_{i1} > 0$, $\gamma_{i2} \geqslant 0$ ($i = 1, 2$), and $\alpha[\psi]$, $\beta[\varphi]$ are two linear functionals on $C[0, 1]$ defined by

$$\alpha[\psi] = \int_0^1 \psi(t)\, \mathrm{d}A(t), \qquad \beta[\varphi] = \int_0^1 \varphi(t)\, \mathrm{d}B(t)$$

involving Riemann–Stieltjes integral; here A and B are functions with positive measures. The coupled integral boundary value problems under consideration are at resonance. This resonance occurs because the associated linear homogeneous system, $-\varphi''(t) = 0$ and $-\psi''(t) = 0$, admits nontrivial solutions when the resonant condition $\alpha[1]\beta[1] = \gamma_{11}\gamma_{21}$ is satisfied. The existence of positive solutions is established using a Leggett–Williams norm-type theorem developed by O'Regan and Zima [18]. The method and theoretical framework employed in this paper are based on [10, 17, 18].

2 Preliminaries

Let us first recall some facts on Fredholm operators and the Leggett–Williams norm-type theorem due to O'Regan and Zima [18].

Definition 1. (See [17].) Let X and Y be real Banach spaces. A linear operator $L : \operatorname{dom} L \subset X \to Y$ is a Fredholm operator of index zero if $\operatorname{Im} L$ is a closed subspace of Y and $\dim\ker L = \operatorname{codim}\operatorname{Im} L < \infty$.

It is well known that for a Fredholm operator L with index zero, there exist two continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that

$$\operatorname{Im} P = \ker L, \qquad X = \ker L \oplus \ker P,$$
$$\operatorname{Im} L = \ker Q, \qquad Y = \operatorname{Im} L \oplus \operatorname{Im} Q,$$

and an isomorphism $J : \operatorname{Im} Q \to \ker L$. Moreover, the operator

$$L_P : \operatorname{dom} L \cap \ker P \to \operatorname{Im} L$$

is invertible, and its inverse is denoted by K_P (the generalized inverse of L_P). Thus, by [17, 20], $Lx = Nx$ is equivalent to

$$x = (P + JQN)x + K_P(I - Q)Nx.$$

Definition 2. (See [8, 11].) Let X be a real Banach space. A nonempty closed convex set C is said to be a cone, provided that

(i) $\lambda x \in C$ for all $x \in C$ and $\lambda \geqslant 0$, and
(ii) $x, -x \in C$ implies $x = \theta$.

It is well known that the cone C induces a partial order on X, defined by

$$x \leqslant y \quad \text{if and only if} \quad y - x \in C.$$

Lemma 1. (*See* [19].) *Let C be a cone in X. Then for every $u \in C \setminus \{\theta\}$, there exists $\sigma(u) > 0$ such that*

$$\|x + u\| \geqslant \sigma(u)\|x\|, \quad x \in C.$$

Let $\Psi = P + JQN + K_P(I - Q)N$ and $\Psi_\gamma = \Psi \circ \gamma$, where $\gamma : X \to C$ is a retraction.

Theorem 1. (*See* [18].) *Let C be a cone in X, and let Ω_1 and Ω_2 be open bounded subsets of X such that $\overline{\Omega}_1 \subset \Omega_2$ and $C \cap (\overline{\Omega}_2 \setminus \Omega_1) \neq \emptyset$. Assume that the following conditions are satisfied*:

(i) *L is a Fredholm operator of index zero*;
(ii) *$QN : X \to Y$ is bounded and continuous, and $K_P(I - Q)N : X \to X$ is compact on every bounded subset of X*;
(iii) *$Lx \neq \lambda Nx$ for all $x \in C \cap \partial\Omega_2 \cap domL$ and $\lambda \in (0, 1)$*;
(iv) *γ maps subsets of Ω_2 into bounded subsets of C*;
(v) *$d_B([I - (P + JQN)\gamma]|_{\ker L}, \ker L \cap \Omega_2, 0) \neq 0$, where d_B stands for the Brouwer degree*;
(vi) *There is a $u_0 \in C \setminus \{\theta\}$ such that for $x \in C(u_0) \cap \partial\Omega_1$, $\|x\| \leqslant \sigma(u_0)\|\Psi x\|$, where $C(u_0) = \{x \in C\colon \mu u_0 \leqslant x$ for some $\mu > 0\}$, and $\sigma(u_0)$ satisfies $\|x + u_0\| \geqslant \sigma(u_0)\|x\|$*;
(vii) *$(P + JQN)\gamma(\partial\Omega_2) \subset C$*;
(viii) *$\Psi_\gamma(\overline{\Omega_2 \setminus \Omega_1}) \subset C$.*

Then the equation $Lx = Nx$ admits a solution in $C \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3 Main result

For simplicity of notation, we set

$$k_i(t,s) = \begin{cases} 1 - s + \frac{\gamma_{i2}}{\gamma_{i1}}, & 0 \leqslant t \leqslant s \leqslant 1, \\ 1 - t + \frac{\gamma_{i2}}{\gamma_{i1}}, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \quad i = 1,2,$$

$$k_1(s) = \int_0^1 k_1(t,s)\,\mathrm{d}t, \qquad k_A(s) = \int_0^1 k_2(t,s)\,\mathrm{d}A(t),$$

$$k_2(s) = \int_0^1 k_2(t,s)\,\mathrm{d}t, \qquad k_B(s) = \int_0^1 k_1(t,s)\,\mathrm{d}B(t),$$

$$G_{11}(t,s) = k_1(t,s) - \frac{\alpha[1]}{2\gamma_{21}(\alpha[1]+\gamma_{11})}k_B(s) + \frac{M\alpha[1]}{2\gamma_{11}\gamma_{21}}k_B(s) - \frac{\alpha[1]}{\alpha[1]+\gamma_{11}}k_1(s),$$

$$G_{12}(t,s) = \frac{1}{2(\alpha[1]+\gamma_{11})}k_A(s) + \frac{M}{2\gamma_{11}}k_A(s) - \frac{\alpha[1]}{\alpha[1]+\gamma_{11}}k_2(s),$$

$$G_{21}(t,s) = \frac{\alpha[1]}{2\gamma_{21}(\alpha[1]+\gamma_{11})}k_B(s) + \frac{M}{2\gamma_{21}}k_B(s) - \frac{\gamma_{11}}{\alpha[1]+\gamma_{11}}k_1(s),$$

$$G_{22}(t,s) = k_2(t,s) + \frac{1}{2(\alpha[1]+\gamma_{11})}k_A(s) + \frac{M\beta[1]}{2\gamma_{11}\gamma_{21}}k_A(s) - \frac{\gamma_{11}}{\alpha[1]+\gamma_{11}}k_2(s).$$

Consider the Banach spaces $X \times X = Y \times Y = C[0,1] \times C[0,1]$ with the norm $\|(u,v)\| = \max\{\|u\|, \|v\|\}$, where $\|u\| = \max_{t\in[0,1]}|u(t)|$. Note that, for $i = 1,2$, $k_i(t,s)$ is the Green's function of the following boundary value problems:

$$-\varphi''(t) = 0, \quad t \in (0,1),$$
$$\varphi'(0) = 0, \qquad \gamma_{i1}\varphi(1) + \gamma_{i2}\varphi'(1) = 0.$$

Then we can associate Eq. (1) with the perturbed Hammerstein integral equations

$$\begin{aligned} \varphi(t) &= \frac{1}{\gamma_{11}}\alpha[\psi] + \int_0^1 k_1(t,s)F_1\big(s,\varphi(s),\psi(s)\big)\,\mathrm{d}s, \\ \psi(t) &= \frac{1}{\gamma_{21}}\beta[\varphi] + \int_0^1 k_2(t,s)F_2\big(s,\varphi(s),\psi(s)\big)\,\mathrm{d}s. \end{aligned} \tag{2}$$

Thus, we can write (2) as a coincidence equation

$$L(\varphi,\psi) = N(\varphi,\psi), \tag{3}$$

where $L : X \times X \to Y \times Y$ is given by

$$L(\varphi,\psi)(t) = \left(\varphi(t) - \frac{1}{\gamma_{11}}\alpha[\psi],\ \psi(t) - \frac{1}{\gamma_{21}}\beta[\varphi]\right), \quad t \in [0,1], \tag{4}$$

and $N : X \times X \to Y \times Y$ is given by

$$N(\varphi,\psi)(t) = \left(\int_0^1 k_1(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,\ \int_0^1 k_2(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\right).$$

To study problem (1), we use the following assumptions.

(H$_1$) Let $F_1, F_2 : [0,1] \times [0,+\infty) \times [0,+\infty) \to \mathbb{R}$ be continuous, and let there exist nonnegative constants a_{ij} ($i = 1,2$, $j = 1,2,3$) such that

$$\bigl|F_1(t,x,y)\bigr| \leqslant a_{11}x + a_{12}y + a_{13}, \qquad \bigl|F_2(t,x,y)\bigr| \leqslant a_{21}x + a_{22}y + a_{23},$$
$$a_{21}a_{12}\gamma_{21} + 2a_{12}\|\beta\| < \gamma_{21}(2 - a_{22})(2 - a_{11}),$$

and

$$a_{21}a_{12}\gamma_{11} + 2a_{21}\|\alpha\| < \gamma_{11}(2 - a_{11})(2 - a_{22}),$$

where $\|\alpha\|$ and $\|\beta\|$ denote the norms of linear functionals α and β, respectively.

(H$_2$) There is a constant $A > 0$ such that

$$F_1(t,x,y) < 0, \quad t \in [0,1],\ x \geqslant A,\ y \in [0,+\infty),$$

and

$$F_2(t,x,y) < 0, \quad t \in [0,1],\ x \in [0,+\infty),\ y \geqslant A.$$

(H$_3$) There exist constants $\delta, M > 0$ such that

$$F_1(t,x,y) > -\delta x, \quad t \in [0,1],\ x,y \in [0,+\infty),$$
$$F_2(t,x,y) > -\delta y, \quad t \in [0,1],\ x,y \in [0,+\infty),$$
$$\frac{\alpha[1]}{\gamma_{11} + \alpha[1]} \geqslant \delta G_{1j}(t,s) \geqslant 0, \qquad \frac{\gamma_{11}}{\gamma_{11} + \alpha[1]} \geqslant \delta G_{2j}(t,s) \geqslant 0, \quad j = 1,2,$$
$$\frac{1}{\gamma_{11} + \alpha[1]} - \frac{M\gamma_{11}}{2\gamma_{11}^2\gamma_{21}}k_B(s) \geqslant 0, \qquad \frac{1}{\gamma_{11} + \alpha[1]} - \frac{M\beta[1]}{2\gamma_{11}^2\gamma_{21}}k_A(s) \geqslant 0,$$
$$1 - \frac{M\delta}{2\gamma_{11}^2\gamma_{21}}\int_0^1 \bigl(\alpha[1]\gamma_{11}k_B(s) + \gamma_{11}\beta[1]k_A(s)\bigr)\,\mathrm{d}s \geqslant 0.$$

(H$_4$) There exist $r \in (0,\infty)$, $t_0 \in [0,1]$, $a \in (0,1]$, $T \in (0,1)$, and continuous functions $g_i : [0,1] \to [0,\infty)$ and $h_i : (0,r] \to [0,\infty)$ such that for all $t \in [0,1]$ and $x, y \in (0,r]$, $F_1(t,x,y) \geqslant g_1(t)h_1(x)$, $F_2(t,x,y) \geqslant g_2(t)h_2(y)$, $h_i(s)/s^a$ ($i = 1,2$) is nonincreasing on $(0,r]$, and

$$\frac{h_1(r)}{r}\int_0^1 G_{i1}(t_0,s)g_1(s)\,\mathrm{d}s + \frac{h_2(r)}{r}\int_0^1 G_{i2}(t_0,s)g_2(s)\,\mathrm{d}s \geqslant \frac{1-T}{T^a}, \quad i = 1,2.$$

Now we begin with the following lemmas to obtain our main results.

Lemma 2. *Let L be defined by* (4). *Then*

(i) $\ker L = \{(\varphi, \psi) = c(\alpha[1], \gamma_{11}),\ c \in \mathbb{R}\}$,
(ii) $\operatorname{Im} L = \{(u, v)\colon\ \beta[1]\alpha[v] + \gamma_{11}\beta[u] = 0\}$.

Proof. Clearly, $L(\varphi, \psi) = (0, 0)$ has the solution $\varphi(t) \equiv \alpha[\psi]/\gamma_{11}$, $\psi(t) \equiv \beta[\varphi]/\gamma_{21}$. Using the boundary condition, we have

$$\varphi(t) = \frac{1}{\gamma_{11}}\alpha[\psi] = \frac{1}{\gamma_{11}}\frac{1}{\gamma_{21}}\alpha\bigl[\beta[\varphi]\bigr] = \frac{1}{\gamma_{11}}\frac{1}{\gamma_{21}}\beta[\varphi]\alpha[1] = \frac{1}{\gamma_{11}}\psi(t)\alpha[1].$$

Thus,

$$\ker L \subset \bigl\{(\varphi, \psi) = c(\alpha[1], \gamma_{11}),\ c \in \mathbb{R}\bigr\}.$$

On the other hand, suppose that $(\varphi, \psi) = (\alpha[1], \gamma_{11})$. Then

$$\begin{aligned} L(\varphi, \psi) &= \left(\varphi(t) - \frac{1}{\gamma_{11}}\alpha[\psi],\ \psi(t) - \frac{1}{\gamma_{21}}\beta[\varphi]\right) \\ &= \left(\alpha[1] - \frac{1}{\gamma_{11}}\alpha[\gamma_{11}],\ \gamma_{11} - \frac{1}{\gamma_{21}}\beta\bigl[\alpha[1]\bigr]\right) \\ &= \left(0,\ \gamma_{11} - \frac{1}{\gamma_{21}}\beta[1]\alpha[1]\right) = (0, 0). \end{aligned}$$

Thus, we conclude that (i) holds, and $\dim\ker L = 1$. Note that if $\alpha[1]\beta[1] \neq \gamma_{11}\gamma_{21}$, then $\ker L = \{\theta\}$.

Next, we will prove (ii). If $(\varphi, \psi) \in \operatorname{dom} L$ and $L(\varphi, \psi) = (u, v)$, then by the definition of L we have

$$\varphi(t) = \frac{1}{\gamma_{11}}\alpha[\psi] + u(t), \qquad \psi(t) = \frac{1}{\gamma_{21}}\beta[\varphi] + v(t).$$

Applying boundary conditions, we obtain

$$\begin{aligned} \beta[\varphi] &= \frac{1}{\gamma_{11}}\alpha[\psi]\beta[1] + \beta[u] = \frac{1}{\gamma_{11}}\left(\frac{1}{\gamma_{21}}\beta[\varphi]\alpha[1] + \alpha[v]\right)\beta[1] + \beta[u] \\ &= \beta[\varphi] + \frac{1}{\gamma_{11}}\alpha[v]\beta[1] + \beta[u]. \end{aligned}$$

Thus,

$$\operatorname{Im} L \subset \bigl\{(u, v)\colon\ \beta[1]\alpha[v] + \gamma_{11}\beta[u] = 0\bigr\}.$$

On the other hand, if (u, v) satisfies $\beta[1]\alpha[v] + \gamma_{11}\beta[u] = 0$, we can construct φ and ψ such that $L(\varphi, \psi) = (u, v)$. Let $\varphi(t) = u(t)$, $\psi(t) = v(t) + \beta[\varphi]/\gamma_{21}$. Then

$$\begin{aligned} \alpha[\psi] &= \alpha[v] + \frac{1}{\gamma_{21}}\beta[\varphi]\alpha[1] = \frac{\alpha[1]\beta[1]}{\gamma_{11}\gamma_{21}}\alpha[v] + \frac{1}{\gamma_{21}}\beta[\varphi]\alpha[1] \\ &= \frac{\alpha[1]}{\gamma_{11}\gamma_{21}}\bigl(\beta[1]\alpha[v] + \gamma_{11}\beta[u]\bigr) = 0. \end{aligned}$$

Thus,

$$L(\varphi,\psi)=\left(\varphi(t)-\frac{1}{\gamma_{11}}\alpha[\psi],\ \psi(t)-\frac{1}{\gamma_{21}}\beta[\varphi]\right)=(\varphi,v)=(u,v),$$

and

$$\bigl\{(u,v)\colon\ \beta[1]\alpha[v]+\gamma_{11}\beta[u]=0\bigr\}\subset \operatorname{Im} L.$$

Hence, (ii) is proved. □

Lemma 3. *Let L be defined by* (4)*. Then L is a Fredholm operator of index zero. Moreover, the linear continuous projector operators $P: X\times X\to X\times X$ and $Q: Y\times Y\to Y\times Y$ can be defined as*

$$\begin{aligned}P(\varphi,\psi)&=\frac{1}{\alpha[1]+\gamma_{11}}\int_0^1\bigl(\varphi(t)+\psi(t)\bigr)\,\mathrm{d}t\cdot\bigl(\alpha[1],\gamma_{11}\bigr),\\ Q(u,v)&=\frac{1}{2\gamma_{11}^2\gamma_{21}}\bigl(\beta[1]\alpha[v]+\gamma_{11}\beta[u]\bigr)\cdot\bigl(\alpha[1],\gamma_{11}\bigr).\end{aligned}\tag{5}$$

Furthermore, the operator K_P, the inverse of L_P, can be written by

$$\begin{aligned}K_p(u,v)(t)=\biggl(&u(t)+\frac{\alpha[v]-\alpha[1]\int_0^1(u(t)+v(t))\,\mathrm{d}t}{\alpha[1]+\gamma_{11}},\\ &v(t)+\frac{\alpha[1]\beta[u]-\gamma_{11}\gamma_{21}\int_0^1(u(t)+v(t))\,\mathrm{d}t}{\gamma_{21}(\alpha[1]+\gamma_{11})}\biggr).\end{aligned}$$

Proof. For $(\varphi,\psi)\in X\times X$ and $(u,v)\in Y\times Y$, we have

$$\begin{aligned}P^2(\varphi,\psi)&=\frac{1}{(\alpha[1]+\gamma_{11})^2}\left(\alpha[1]\int_0^1\bigl(\varphi(t)+\psi(t)\bigr)\,\mathrm{d}t+\gamma_{11}\int_0^1\bigl(\varphi(t)+\psi(t)\bigr)\,\mathrm{d}t\right)\\&\quad\times(\alpha[1],\gamma_{11})\\&=\frac{1}{\alpha[1]+\gamma_{11}}\int_0^1\bigl(\varphi(t)+\psi(t)\bigr)\,\mathrm{d}t\cdot\bigl(\alpha[1],\gamma_{11}\bigr)=P(\varphi,\psi),\quad(\varphi,\psi)\in X\times X,\\ Q^2(u,v)&=\frac{1}{4\gamma_{11}^4\gamma_{21}^2}\bigl(\beta[1]\alpha[v]+\gamma_{11}\beta[u]\bigr)\bigl(\beta[1]\alpha[\gamma_{11}]+\gamma_{11}\beta[\alpha[1]]\bigr)\bigl(\alpha[1],\gamma_{11}\bigr)\\&=\frac{1}{4\gamma_{11}^4\gamma_{21}^2}\bigl(\beta[1]\alpha[v]+\gamma_{11}\beta[u]\bigr)\bigl(\gamma_{11}\beta[1]\alpha[1]+\gamma_{11}\alpha[1]\beta[1]\bigr)\bigl(\alpha[1],\gamma_{11}\bigr)\\&=\frac{1}{2\gamma_{11}^2\gamma_{21}}\bigl(\beta[1]\alpha[v]+\gamma_{11}\beta[u]\bigr)\bigl(\alpha[1],\gamma_{11}\bigr)=Q(u,v),\quad(u,v)\in Y\times Y.\end{aligned}$$

Then P and Q are projection operators. Also, $\ker Q = \operatorname{Im} L$ and $\operatorname{Im} P = \ker L$. Since Q is a projection operator, we obtain $Y \times Y = \operatorname{Im} Q \oplus \ker Q = \operatorname{Im} Q \oplus \operatorname{Im} L$, and $\dim \ker L = \operatorname{codim} \operatorname{Im} L = 1$. Therefore, L is a Fredholm operator of index zero.

Let L_P be the restriction of the operator L to $\ker P \cap \operatorname{dom} L$. Take

$$u = \varphi - \frac{1}{\gamma_{11}}\alpha[\psi], \qquad v = \psi - \frac{1}{\gamma_{21}}\beta[\varphi]$$

for $(\varphi, \psi) \in \ker P = \{(\varphi, \psi) \in X \times X\colon \int_0^1 (\varphi(t) + \psi(t))\,\mathrm{d}t = 0\}$. Then

$$\begin{aligned} 0 &= \int\limits_0^1 \bigl(\varphi(t) + \psi(t)\bigr)\,\mathrm{d}t = \frac{1}{\gamma_{11}}\alpha[\psi] + \frac{1}{\gamma_{21}}\beta[\varphi] + \int\limits_0^1 \bigl(u(t) + v(t)\bigr)\,\mathrm{d}t \\ &= \frac{1}{\gamma_{11}}\alpha[v] + \frac{1}{\gamma_{11}\gamma_{21}}\beta[\varphi]\alpha[1] + \frac{1}{\gamma_{21}}\beta[\varphi] + \int\limits_0^1 \bigl(u(t) + v(t)\bigr)\,\mathrm{d}t. \end{aligned}$$

This gives

$$\begin{aligned} \beta[\varphi] &= -\frac{\gamma_{21}\alpha[v] + \gamma_{11}\gamma_{21}\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\alpha[1] + \gamma_{11}} \\ &= \frac{\alpha[1]\beta[u] - \gamma_{11}\gamma_{21}\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\alpha[1] + \gamma_{11}} \end{aligned}$$

and

$$\begin{aligned} \alpha[\psi] &= \alpha[v] + \frac{1}{\gamma_{21}}\beta[\varphi]\alpha[1] \\ &= \alpha[v] + \frac{\alpha[1]}{\gamma_{21}}\frac{\alpha[1]\beta[u] - \gamma_{11}\gamma_{21}\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\alpha[1] + \gamma_{11}} \\ &= \frac{\gamma_{11}\alpha[v] - \alpha[1]\gamma_{11}\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\alpha[1] + \gamma_{11}}. \end{aligned}$$

Thus, for $(u, v) \in \operatorname{Im} L$, the inverse K_P of L_P is given by

$$\begin{aligned} K_p(u, v)(t) = \biggl(&u(t) + \frac{\alpha[v] - \alpha[1]\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\alpha[1] + \gamma_{11}}, \\ &v(t) + \frac{\alpha[1]\beta[u] - \gamma_{11}\gamma_{21}\int_0^1 (u(t) + v(t))\,\mathrm{d}t}{\gamma_{21}(\alpha[1] + \gamma_{11})}\biggr). \end{aligned}$$ □

For $c \in \mathbb{R}$, define

$$J\bigl[c(\alpha[1], \gamma_{11})\bigr] = cM\bigl(\alpha[1], \gamma_{11}\bigr),$$

where M is a positive constant given in (H_3). Obviously, $J : \operatorname{Im} Q \to \ker L$ is an isomorphism.

Now, coincidence equation (3), that is,

$$\varphi(t) = \frac{1}{\gamma_{11}}\alpha[\psi] + \int_0^1 k_1(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,$$
$$\psi(t) = \frac{1}{\gamma_{21}}\beta[\varphi] + \int_0^1 k_2(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,$$

is equivalent to

$$(\varphi,\psi) = \Psi(\varphi,\psi) = (P+JQN)(\varphi,\psi) + K_P(I-Q)N(\varphi,\psi),$$

that is,

$$\begin{aligned}
\varphi(t) &= \frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int_0^1 \varphi(s)\,\mathrm{d}s + \frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int_0^1 \psi(s)\,\mathrm{d}s \\
&\quad + \int_0^1 G_{11}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s + \int_0^1 G_{12}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s, \\
\psi(t) &= \frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int_0^1 \varphi(s)\,\mathrm{d}s + \frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int_0^1 \psi(s)\,\mathrm{d}s \\
&\quad + \int_0^1 G_{21}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s + \int_0^1 G_{22}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s.
\end{aligned} \tag{6}$$

Also, from the above proof we obtain

$$\begin{aligned}
&(P+JQN)(\varphi,\psi) \\
&\quad = \bigl(\alpha[1],\gamma_{11}\bigr)\Biggl(\frac{1}{\gamma_{11}+\alpha[1]}\int_0^1 \bigl(\varphi(s)+\psi(s)\bigr)\,\mathrm{d}s \\
&\qquad + \frac{M}{2\gamma_{11}^2\gamma_{21}}\int_0^1 \bigl(\gamma_{11}k_B(s)F_1\bigl(s,\varphi(s),\psi(s)\bigr) + \beta[1]k_A(s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\bigr)\,\mathrm{d}s\Biggr) \\
&\quad = \Biggl(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int_0^1 \bigl(\varphi(s)+\psi(s)\bigr)\,\mathrm{d}s + \frac{M}{2\gamma_{11}}\int_0^1 k_A(s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s \\
&\qquad + \frac{M\alpha[1]}{2\gamma_{11}\gamma_{21}}\int_0^1 k_B(s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,
\end{aligned}$$

$$\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int_0^1 \bigl(\varphi(s)+\psi(s)\bigr)\,\mathrm{d}s+\frac{M\beta[1]}{2\gamma_{11}\gamma_{21}}\int_0^1 k_A(s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s$$
$$+\frac{M}{2\gamma_{21}}\int_0^1 k_B(s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\Biggr).$$

Using the Arzelà–Ascoli theorem, we obtain the following lemmas without proof.

Lemma 4. *$QN : X\times X\to Y\times Y$ is bounded and continuous, and $K_P(I-Q)N : X\times X\to X\times X$ is compact on every bounded subset of $X\times X$.*

Lemma 5. *Under hypotheses* (H_1) *and* (H_2)*, the set*

$$\Omega=\bigl\{(\varphi,\psi)\in C\cap\operatorname{dom}L\colon\ L(\varphi,\psi)=\lambda N(\varphi,\psi),\ \lambda\in(0,1)\bigr\}$$

is bounded.

Proof. Take $(\varphi,\psi)\in\Omega$, then $N(\varphi,\psi)\in\operatorname{Im}L=\ker Q$. By (5), we have

$$\beta[1]\int_0^1 F_2\bigl(t,\varphi(t),\psi(t)\bigr)\,\mathrm{d}A(t)+\gamma_{11}\int_0^1 F_1\bigl(t,\varphi(t),\psi(t)\bigr)\,\mathrm{d}B(t)=0. \tag{7}$$

Subsequently, we discuss two cases.

Case 1. There exists $t_0\in[0,1]$ such that $F_1(t_0,\varphi(t_0),\psi(t_0))\geqslant 0$. So, by ($\mathrm{H}_2$), we have $0\leqslant\varphi(t_0)\leqslant A$. By (1), we obtain

$$\varphi'(t)=\varphi'(0)-\int_0^t F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s=-\int_0^t F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,$$
$$\varphi(t)=\varphi(t_0)-\int_{t_0}^t\int_0^\tau F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\,\mathrm{d}\tau.$$

This, together with (H_1), implies that

$$\begin{aligned}\varphi(t)&\leqslant\varphi(t_0)+\int_0^1\int_0^\tau\bigl|F_1\bigl(s,\varphi(s),\psi(s)\bigr)\bigr|\,\mathrm{d}s\,\mathrm{d}\tau\\&\leqslant A+\bigl(a_{11}\|\varphi\|+a_{12}\|\psi\|+a_{13}\bigr)\int_0^1\int_0^\tau\mathrm{d}s\,\mathrm{d}\tau\\&=\frac{a_{11}}{2}\|\varphi\|+\frac{a_{12}}{2}\|\psi\|+\biggl(\frac{a_{13}}{2}+A\biggr).\end{aligned}$$

Consequently, we have

$$\|\varphi\|\leqslant\frac{2A+a_{13}}{2-a_{11}}+\frac{a_{12}}{2-a_{11}}\|\psi\|. \tag{8}$$

In view of (2), we can find that

$$
\begin{aligned}
\psi(t) &= \int_0^1 k_2(t,s) F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s + \frac{1}{\gamma_{21}}\beta[\varphi] \\
&\leqslant \bigl(a_{21}\|\varphi\| + a_{22}\|\psi\| + a_{23}\bigr) \int_0^1 k_2(t,s)\,\mathrm{d}s + \frac{\|\beta\|}{\gamma_{21}}\|\varphi\| \\
&\leqslant \left(\frac{a_{21}}{2} + \frac{\|\beta\|}{\gamma_{21}}\right)\|\varphi\| + \frac{a_{22}}{2}\|\psi\| + \frac{a_{23}}{2} \\
&\leqslant \frac{a_{21}a_{12}\gamma_{21} + 2a_{12}\|\beta\| + a_{22}\gamma_{21}(2-a_{11})}{2\gamma_{21}(2-a_{11})}\|\psi\| + D_1,
\end{aligned}
$$

where

$$
D_1 = \frac{a_{23}}{2} + \frac{(a_{21}\gamma_{21} + 2\|\beta\|)(2A + a_{13})}{2\gamma_{21}(2-a_{11})}.
$$

This leads to

$$
\|\psi\| \leqslant \frac{2D_1\gamma_{21}(2-a_{11})}{\gamma_{21}(2-a_{22})(2-a_{11}) - a_{21}a_{12}\gamma_{21} - 2a_{12}\|\beta\|} := D_3.
$$

This, together with (8), implies that Ω is bounded.

Case 2. $F_1(t,\varphi(t),\psi(t)) < 0$ for all $t \in [0,1]$. Then (7) implies that there exists $t_0 \in [0,1]$ such that $F_2(t_0,\varphi(t_0),\psi(t_0)) \geqslant 0$. So, by ($H_2$), we have $0 \leqslant \psi(t_0) \leqslant A$. Similar to the proof of Case 1, we have

$$
\begin{aligned}
\psi(t) &\leqslant \psi(t_0) + \int_0^1 \int_0^\tau \bigl|F_2\bigl(s,\varphi(s),\psi(s)\bigr)\bigr|\,\mathrm{d}s\,\mathrm{d}\tau \\
&\leqslant \frac{a_{21}}{2}\|\varphi\| + \frac{a_{22}}{2}\|\psi\| + \left(A + \frac{a_{23}}{2}\right).
\end{aligned}
$$

Consequently, we obtain

$$
\|\psi\| \leqslant \frac{2A + a_{23}}{2-a_{22}} + \frac{a_{21}}{2-a_{22}}\|\varphi\|.
$$

In view of (2), we can find that

$$
\begin{aligned}
\varphi(t) &= \int_0^1 k_1(t,s) F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s + \frac{1}{\gamma_{11}}\alpha[\psi] \\
&\leqslant \bigl(a_{11}\|\varphi\| + a_{12}\|\psi\| + a_{13}\bigr) \int_0^1 k_1(t,s)\,\mathrm{d}s + \frac{\|\alpha\|}{\gamma_{11}}\|\psi\|
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant \frac{a_{11}}{2}\|\varphi\| + \left(\frac{a_{12}}{2} + \frac{\|\alpha\|}{\gamma_{11}}\right)\|\psi\| + \frac{a_{13}}{2} \\
&\leqslant \frac{a_{12}a_{21}\gamma_{11} + 2a_{21}\|\alpha\| + a_{11}\gamma_{11}(2-a_{22})}{2\gamma_{11}(2-a_{22})}\|\varphi\| + D_2,
\end{aligned}
$$

where

$$
D_2 = \frac{a_{13}}{2} + \frac{(a_{12}\gamma_{11} + 2\|\alpha\|)(2A + a_{23})}{2\gamma_{11}(2-a_{22})}.
$$

This leads to

$$
\|\varphi\| \leqslant \frac{2D_2\gamma_{11}(2-a_{22})}{\gamma_{11}(2-a_{11})(2-a_{22}) - a_{21}a_{12}\gamma_{11} - 2a_{21}\|\alpha\|} := D_4.
$$

Hence, it follows from (8) that Ω is bounded. □

We now prove the main result of this article.

Theorem 2. *Suppose that conditions* (H$_1$)–(H$_4$) *hold. Then problem* (1) *has at least one positive solution in* $X \times X$.

Proof. Consider the cone of nonnegative function in $X \times X$

$$
C = \big\{(\varphi,\psi) \in X \times X\colon\ \varphi(t) \geqslant 0,\ \psi(t) \geqslant 0,\ t \in [0,1]\big\}.
$$

Let

$$
\begin{aligned}
\Omega_1 &= \big\{(\varphi,\psi) \in X \times X\colon\ T\big\|(\varphi,\psi)\big\| < \big|\varphi(t)\big|,\ \big|\psi(t)\big| < r,\ t \in [0,1]\big\}, \\
\Omega_2 &= \big\{(\varphi,\psi) \in X \times X\colon\ \big\|(\varphi,\psi)\big\| < R\big\},
\end{aligned}
$$

where

$$
R = \max\left\{D_3, D_4, \frac{A\max\{\alpha[1],\gamma_{11}\}}{\alpha[1]}, \frac{A\max\{\alpha[1],\gamma_{11}\}}{\gamma_{11}}\right\} + r,
$$

T is a positive constant given in (H$_4$). Clearly, Ω_1 and Ω_2 are open bounded subsets of $X \times X$, and $\overline{\Omega_1} \subset \Omega_2$. Define $(\gamma(\varphi,\psi))(t) = (|\varphi(t)|, |\psi(t)|)$ for $(\varphi,\psi) \in X \times X$. Then $\gamma: X \times X \to C$ is a retraction that maps bounded subsets of $\overline{\Omega_2}$ into bounded subsets of C. Consequently, (iv) holds. From Lemmas 2–5 it follows that assumptions (i)–(iii) of Theorem 1 are fulfilled.

To prove (v), we consider $(\varphi,\psi) \in \ker L \cap \overline{\Omega_2}$. Then $(\varphi,\psi) = c(\alpha[1],\gamma_{11})$ with $c \in [-\overline{R},\overline{R}]$, where $\overline{R} = R/\max\{\alpha[1],\gamma_{11}\}$. For $c \in [-\overline{R},\overline{R}]$ and $\lambda \in [0,1]$, consider

$$
\begin{aligned}
H(c,\lambda) &= \big[I - \lambda(P + JQN)\gamma\big]\big(c\alpha[1], c\gamma_{11}\big) \\
&= \left(c - \lambda\left(|c| + \frac{M}{2\gamma_{11}^2\gamma_{21}}\int_0^1 \big(\gamma_{11}k_B(s)F_1\big(s,\alpha[1]|c|,\gamma_{11}|c|\big)\right.\right. \\
&\qquad \left.\left. + \beta[1]k_A(s)F_2\big(s,\alpha[1]|c|,\gamma_{11}|c|\big)\big)\,\mathrm{d}s\right)\right)\big(\alpha[1],\gamma_{11}\big).
\end{aligned}
$$

Assume that $H(c,\lambda)=0$ for $(\varphi,\psi)=c(\alpha[1],\gamma_{11})\in\partial\Omega_2$. Then, according to ($H_3$),

$$\begin{aligned}c&=\lambda\Biggl(|c|+\frac{M}{2\gamma_{11}^2\gamma_{21}}\int\limits_0^1\bigl(\gamma_{11}k_B(s)F_1\bigl(s,\alpha[1]|c|,\gamma_{11}|c|\bigr)\\&\qquad+\beta[1]k_A(s)F_2\bigl(s,\alpha[1]|c|,\gamma_{11}|c|\bigr)\bigr)\,\mathrm{d}s\Biggr)\\&\geqslant\lambda|c|\Biggl(1-\frac{M\delta}{2\gamma_{11}^2\gamma_{21}}\int\limits_0^1\bigl(\alpha[1]\gamma_{11}k_B(s)+\gamma_{11}\beta[1]k_A(s)\bigr)\,\mathrm{d}s\Biggr)\geqslant 0.\end{aligned}$$

Hence, $H(c,\lambda)=0$ gives $c\geqslant 0$. Since $(\varphi,\psi)=c(\alpha[1],\gamma_{11})\in\partial\Omega_2$, we have $|c|=\overline{R}$; and, together with $c\geqslant 0$, this implies $c=\overline{R}$. Furthermore, we conclude that $\lambda\neq 0$. In this case, noting that $\alpha[1]\overline{R}>A$ and $\gamma_{11}\overline{R}>A$, we would have by ($H_3$)

$$\begin{aligned}0&\leqslant\overline{R}(1-\lambda)\\&=\frac{M\lambda}{2\gamma_{11}^2\gamma_{21}}\Biggl(\int\limits_0^1\gamma_{11}k_B(s)F_1\bigl(s,\alpha[1]\overline{R},\gamma_{11}\overline{R}\bigr)\,\mathrm{d}s+\int\limits_0^1\beta[1]k_A(s)F_2\bigl(s,\alpha[1]\overline{R},\gamma_{11}\overline{R}\bigr)\,\mathrm{d}s\Biggr)\\&<0,\end{aligned}$$

which is impossible. So, $H(c,\lambda)\neq 0$ for $(\varphi,\psi)=c(\alpha[1],\gamma_{11})\in\partial\Omega_2$ and $\lambda\in[0,1]$. Therefore, by the homotopy invariance property of topological degree, we have

$$d_B\bigl(\bigl[I-(P+JQN)\gamma\bigr]\big|_{\ker L},\,\ker L\cap\Omega_2,\,\theta\bigr)=d_B(I,\,\ker L\cap\Omega_2,\,\theta)\neq 0,$$

which shows that (v) of Theorem 1 holds.

Let $(\varphi,\psi)\in\overline{\Omega_2}\setminus\Omega_1$. By (6), we conclude that

$$\begin{aligned}&\bigl(\Psi_\gamma(\varphi,\psi)\bigr)(t)\\&\quad=\bigl(\bigl[\bigl(P+JQN+K_P(I-Q)N\bigr)\circ\gamma\bigr](\varphi,\psi)\bigr)(t)\\&\quad=\Biggl(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int\limits_0^1\varphi(s)\,\mathrm{d}s+\frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int\limits_0^1\psi(s)\,\mathrm{d}s\\&\qquad+\int\limits_0^1G_{11}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s+\int\limits_0^1G_{12}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,\\&\qquad\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int\limits_0^1\varphi(s)\,\mathrm{d}s+\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int\limits_0^1\psi(s)\,\mathrm{d}s\\&\qquad+\int\limits_0^1G_{21}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s+\int\limits_0^1G_{22}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\Biggr)\end{aligned}$$

$$\geqslant \left(\int_0^1 \left(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]} - \delta G_{11}(t,s) \right) \left|\varphi(s)\right| \mathrm{d}s + \int_0^1 \left(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]} - \delta G_{12}(t,s) \right) \left|\psi(s)\right| \mathrm{d}s, \right.$$
$$\left. \int_0^1 \left(\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} - \delta G_{21}(t,s) \right) \left|\varphi(s)\right| \mathrm{d}s + \int_0^1 \left(\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} - \delta G_{22}(t,s) \right) \left|\psi(s)\right| \mathrm{d}s \right)$$
$$\geqslant (0,0).$$

which implies (viii). By (H_1) and (H_4), for $(\varphi,\psi) \in \partial\Omega_2$, we have

$$\begin{aligned}
&(P + JQN)\gamma(\varphi,\psi) \\
&\quad = \left(\frac{M}{2\gamma_{11}^2\gamma_{21}} \int_0^1 \left(\gamma_{11} k_B(s) F_1\left(s,\varphi(s),\psi(s)\right) + \beta[1] k_A(s) F_2\left(s,\varphi(s),\psi(s)\right)\right) \mathrm{d}s \right. \\
&\qquad \left. + \frac{1}{\gamma_{11}+\alpha[1]} \int_0^1 \left(\varphi(s)+\psi(s)\right) \mathrm{d}s \right) \left(\alpha[1],\gamma_{11}\right) \\
&\quad \geqslant \left(\int_0^1 \left(\frac{1}{\gamma_{11}+\alpha[1]} - \frac{M\gamma_{11}}{2\gamma_{11}^2\gamma_{21}} k_B(s) \right) \left|\varphi(s)\right| \mathrm{d}s \right. \\
&\qquad \left. + \int_0^1 \left(\frac{1}{\gamma_{11}+\alpha[1]} - \frac{M\beta[1]}{2\gamma_{11}^2\gamma_{21}} k_A(s) \right) \left|\psi(s)\right| \mathrm{d}s \right) \left(\alpha[1],\gamma_{11}\right) \\
&\quad \geqslant (0,0).
\end{aligned}$$

This means that (vii) holds.

It remains to show that (vi) is satisfied. Let $(\varphi_0,\psi_0) = (1,1)$, then we have $(\varphi_0,\psi_0) \in C \setminus \{\theta\}$, $C(\varphi_0,\psi_0) = \{(\varphi,\psi) |\ \varphi(t) > 0, \psi(t) > 0\}$, and we take $\sigma(\varphi_0,\psi_0) = 1$. Let $(\varphi,\psi) \in C(\varphi_0,\psi_0) \cap \partial\Omega_1$. Then $T\|(\varphi,\psi)\| \leqslant \varphi(t),\psi(t) \leqslant r$, $t \in [0,1]$. Therefore, combining with (H_3) and (H_4), we get

$$\begin{aligned}
&\frac{\alpha[1]}{\gamma_{11}+\alpha[1]} \int_0^1 \varphi(s)\,\mathrm{d}s + \frac{\alpha[1]}{\gamma_{11}+\alpha[1]} \int_0^1 \psi(s)\,\mathrm{d}s \\
&\quad \geqslant T\left(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]} \int_0^1 \mathrm{d}s + \frac{\alpha[1]}{\gamma_{11}+\alpha[1]} \int_0^1 \mathrm{d}s \right) \left\|(\varphi,\psi)\right\| = \frac{2T\alpha[1]}{\gamma_{11}+\alpha[1]} \left\|(\varphi,\psi)\right\|, \\
&\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} \int_0^1 \varphi(s)\,\mathrm{d}s + \frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} \int_0^1 \psi(s)\,\mathrm{d}s \\
&\quad \geqslant T\left(\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} \int_0^1 \mathrm{d}s + \frac{\gamma_{11}}{\gamma_{11}+\alpha[1]} \int_0^1 \mathrm{d}s \right) \left\|(\varphi,\psi)\right\| = \frac{2T\gamma_{11}}{\gamma_{11}+\alpha[1]} \left\|(\varphi,\psi)\right\|,
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^1 G_{i1}(t_0,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{i2}(t_0,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\\
&\quad\geqslant\int_0^1 G_{i1}\bigl(t_0,s)g_1(s)h_1\bigl(\varphi(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{i2}(t_0,s)g_2(s)h_2\bigl(\psi(s)\bigr)\,\mathrm{d}s\\
&\quad\geqslant\int_0^1 G_{i1}(t_0,s)g_1(s)\frac{h_1(\varphi(s))}{\varphi^a(s)}\varphi^a(s)\,\mathrm{d}s+\int_0^1 G_{i2}(t_0,s)g_2(s)\frac{h_2(\psi(s))}{\psi^a(s)}\psi^a(s)\,\mathrm{d}s\\
&\quad\geqslant\frac{h_1(r)}{r^a}\int_0^1 G_{i1}(t_0,s)g_1(s)T^a\bigl\|(\varphi,\psi)\bigr\|^a\,\mathrm{d}s\\
&\qquad+\frac{h_2(r)}{r^a}\int_0^1 G_{i2}(t_0,s)g_2(s)T^a\bigl\|(\varphi,\psi)\bigr\|^a\,\mathrm{d}s\\
&\quad\geqslant\left(\frac{h_1(r)}{r}\int_0^1 G_{i1}(t_0,s)g_1(s)\,\mathrm{d}s+\frac{h_2(r)}{r}\int_0^1 G_{i2}(t_0,s)g_2(s)\,\mathrm{d}s\right)T^a\bigl\|(\varphi,\psi)\bigr\|\\
&\quad\geqslant(1-T)\bigl\|(\varphi,\psi)\bigr\|,\quad i=1,2,
\end{aligned}
$$

and

$$
\begin{aligned}
&\bigl(\Psi_\gamma(\varphi,\psi)\bigr)(t_0)\\
&\quad=\bigl(\bigl[\bigl(P+JQN+K_P(I-Q)N\bigr)\circ\gamma\bigr](\varphi,\psi)\bigr)(t_0)\\
&\quad=\Biggl(\frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int_0^1\varphi(s)\,\mathrm{d}s+\frac{\alpha[1]}{\gamma_{11}+\alpha[1]}\int_0^1\psi(s)\,\mathrm{d}s\\
&\qquad+\int_0^1 G_{11}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{12}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s,\\
&\qquad\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int_0^1\varphi(s)\,\mathrm{d}s+\frac{\gamma_{11}}{\gamma_{11}+\alpha[1]}\int_0^1\psi(s)\,\mathrm{d}s\\
&\qquad+\int_0^1 G_{21}(t,s)F_1\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s+\int_0^1 G_{22}(t,s)F_2\bigl(s,\varphi(s),\psi(s)\bigr)\,\mathrm{d}s\Biggr)\\
&\quad\geqslant\left(\frac{2T\alpha[1]}{\gamma_{11}+\alpha[1]}\bigl\|(\varphi,\psi)\bigr\|+(1-T)\bigl\|(\varphi,\psi)\bigr\|,\ \frac{2T\gamma_{11}}{\gamma_{11}+\alpha[1]}\bigl\|(\varphi,\psi)\bigr\|+(1-T)\bigl\|(\varphi,\psi)\bigr\|\right).
\end{aligned}
$$

This implies $\|(\varphi,\psi)\| \leqslant \sigma(\varphi_0,\psi_0)\|\Psi_\gamma(\varphi,\psi)\|$ for $(\varphi,\psi) \in C(\varphi_0,\psi_0) \cap \partial\Omega_1$. So (vi) is satisfied, and the assertion follows. $\square$

4 An example

Consider the following coupled integral boundary value problem:

$$\begin{gathered} -\varphi''(t) = F_1\bigl(t,\varphi(t),\psi(t)\bigr), \quad -\psi''(t) = F_2\bigl(t,\varphi(t),\psi(t)\bigr), \quad t \in (0,1), \\ \varphi'(0) = 0, \quad \varphi(1) = \frac{7}{8}\int_0^1 \psi(t)\,\mathrm{d}t, \qquad \psi'(0) = 0, \quad \psi(1) = \frac{8}{7}\int_0^1 \varphi(t)\,\mathrm{d}t \end{gathered} \tag{9}$$

with

$$F_1(t,\varphi,\psi) = \frac{7}{5}\sin\frac{\pi\varphi}{2} - \frac{\varphi}{5} + \frac{\psi}{1+\psi^2}(1+t)$$

and

$$\begin{aligned} &F_2(t,\varphi,\psi) \\ &\quad = \begin{cases} \sqrt{\psi+2}(\psi^2 - 2\psi + 3) + \arctan\varphi + \mathrm{e}^{-\varphi}, & \psi \in [0,1],\ \varphi \in [0,+\infty), \\ \sqrt{\psi+1}(2 - \frac{1}{5}\sqrt{\psi-1}) + \arctan\varphi + \mathrm{e}^{-\varphi}, & \psi \in (1,+\infty),\ \varphi \in [0,+\infty). \end{cases} \end{aligned}$$

Problem (9) has at least one positive solution.
To verify this claim, we compute the following quantities:

$$\begin{gathered} k_1(t,s) = k_2(t,s) = \begin{cases} 1-s, & 0 \leqslant t \leqslant s \leqslant 1, \\ 1-t, & 0 \leqslant s \leqslant t \leqslant 1, \end{cases} \\ \gamma_{11} = \gamma_{21} = 1, \quad \gamma_{12} = \gamma_{22} = 0, \qquad A(t) = \frac{7}{8}t, \quad B(t) = \frac{8}{7}t, \\ k_1(s) = k_2(s) = \frac{1-s^2}{2}, \qquad k_A(s) = \frac{7}{8}k_1(s), \qquad k_B(s) = \frac{8}{7}k_1(s), \\ G_{11}(t,s) = k_1(t,s) + \left(\frac{M}{2} - \frac{11}{15}\right)k_1(s), \qquad G_{12}(t,s) = \left(\frac{7M}{16} - \frac{7}{30}\right)k_1(s), \\ G_{21}(t,s) = \left(\frac{2M}{7} - \frac{4}{15}\right)k_1(s), \qquad G_{22}(t,s) = k_2(t,s) + \left(\frac{M}{2} - \frac{3}{10}\right)k_1(s) \end{gathered}$$

and

$$1 - \frac{M\delta}{2\gamma_{11}^2\gamma_{21}}\int_0^1 \bigl(\alpha[1]\gamma_{11}k_B(s) + \gamma_{11}\beta[1]k_A(s)\bigr)\,\mathrm{d}s = 1 - \frac{M\delta}{3}.$$

Let $a_{11} = a_{22} = 1/5$, $a_{12} = a_{21} = 0$, $a_{13} = a_{23} = 15$, $A = 225$. Then we have

$$\begin{gathered} F_1(t,\varphi,\psi,y) < 0, \quad t \in [0,1],\ \varphi \geqslant A,\ \psi \in [0,+\infty), \\ F_2(t,\varphi,\psi) < 0, \quad t \in [0,1],\ \varphi \in [0,+\infty),\ \psi \geqslant A, \end{gathered}$$

and

$$\left|F_1(t,\varphi,\psi)\right| \leqslant a_{11}\varphi + a_{12}\psi + a_{13},$$
$$\left|F_2(t,\varphi,\psi)\right| \leqslant a_{21}\varphi + a_{22}\psi + a_{23}.$$

Thus, (H_1) and (H_2) are satisfied.

Taking $M = 5/3$ and $\delta = 2/5$, we have

$$F_1(t,\varphi,\psi) > -\delta\varphi, \quad t \in [0,1],\ \varphi,\psi \in [0,+\infty),$$
$$F_2(t,\varphi,\psi) > -\delta\psi, \quad t \in [0,1],\ \varphi,\psi \in [0,+\infty),$$

and

$$1 - \frac{M\delta}{2\gamma_{11}^2\gamma_{21}} \int_0^1 \big(\alpha[1]\gamma_{11}k_B(s) + \gamma_{11}\beta[1]k_A(s)\big)\,\mathrm{d}s = \frac{7}{9} > 0.$$

Therefore, condition (H_3) of Theorem 2 holds.

Let $r = 1$, $a = 1$, $T = 1/2$, $h_1(\varphi) = (6/5)\sin(\pi\varphi/2)$, $h_2(\psi) = 2\sqrt{\psi+2}$, $g_1(t) = g_2(t) = 1$. So, as $\varphi,\psi \in (0,r]$,

$$F_1(t,\varphi,\psi) \geqslant g_1(t)h_1(\varphi), \qquad F_2(t,\varphi,\psi) \geqslant g_2(t)h_2(\psi).$$

Therefore,

$$\begin{aligned}
&\frac{h_1(r)}{r}\int_0^1 G_{11}(t_0,s)g_1(s)\,\mathrm{d}s + \frac{h_2(r)}{r}\int_0^1 G_{12}(t_0,s)g_2(s)\,\mathrm{d}s \\
&\quad = \frac{h_1(r)}{r}\frac{8}{15} + \frac{h_2(r)}{r}\frac{119}{720} = \frac{6}{5}\frac{8}{15} + 2\sqrt{3}\frac{119}{720} \approx 1.212 \geqslant \frac{1-T}{T^a},
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\frac{h_1(r)}{r}\int_0^1 G_{21}(t_0,s)g_1(s)\,\mathrm{d}s + \frac{h_2(r)}{r}\int_0^1 G_{22}(t_0,s)g_2(s)\,\mathrm{d}s \\
&\quad = \frac{h_1(r)}{r}\frac{22}{415} + \frac{h_2(r)}{r}\frac{61}{90}\frac{6}{5}\frac{22}{415} + 2\sqrt{3}\frac{61}{90} \approx 2.392 \geqslant \frac{1-T}{T^a}.
\end{aligned} \tag{11}$$

From (10) and (11) we obtain that condition (H_4) holds. By Theorem 2, problem (9) has at least one positive solution.

5 Conclusions

This paper investigates a resonant system of nonlinear differential equations subject to coupled Riemann–Stieltjes integral boundary conditions. Through the application of the Leggett–Williams norm-type theorem for coincidence equations, a new existence result is obtained. In the future, we intend to study the existence of multiple positive solutions for resonant coupled integral boundary value problems.

Author contributions. The authors (Y.D. and Y.C.) have contributed as follows: methodology, Y.C.; formal analysis, Y.D. and Y.C.; validation, Y.C.; writing – original draft preparation, Y.D. and Y.C.; writing – review & editing, Y.C. Both authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors sincerely thank the reviewer for careful reading and useful comments that have led to the present improved version of the original paper.

References

1. B. Ahmad, J. Henderson, R. Luca, *Boundary Value Problems for Fractional Differential Equations and Systems*, World Scientific, Singapore, 2021, `https://doi.org/10.1142/11942`.
2. H. Amann, Parabolic evolution equations with nonlinear boundary conditions, in *Proceedings of Symposia in Pure Mathematics, Vol. 45*, AMS, Providence, RI, 1986, pp. 17–27, `https://doi.org/10.1090/pspum/045.1/843545`.
3. H. Amann, Parabolic evolution equations and nonlinear boundary conditions, *J. Differ. Equations*, **72**(2):201–269, 1988, `https://doi.org/10.1016/0022-0396(88)90156-8`.
4. D.G. Aronson, A comparison method for stability analysis of nonlinear parabolic problems, *SIAM Rev.*, **20**(2):245–264, 1978, `https://doi.org/10.1137/1020038`.
5. N.A. Asif, R.A. Khan, Positive solutions to singular system with four-point coupled boundary conditions, *J. Math. Anal. Appl.*, **386**(2):848–861, 2012, `https://doi.org/10.1016/j.jmaa.2011.08.039`.
6. Y. Cui, Existence of solutions for coupled integral boundary value problem at resonance, *Publ. Math. Debr.*, **89**:73–88, 2016, `https://doi.org/10.5486/PMD.2016.7312`.
7. Y. Cui, J. Sun, On existence of positive solutions of coupled integral boundary value problems for a nonlinear singular superlinear differential system, *Electron. J. Qual. Theory Differ. Equ.*, **41**:1–13, 2012, `https://doi.org/10.14232/ejqtde.2012.1.41`.
8. K. Deimling, *Nonlinear Functional Analysis*, Springer, New York, 1985, `https://doi.org/10.1007/978-3-662-00547-7`.
9. K. Deng, Global existence and blow-up for a system of heat equations with nonlinear boundary condition, *Math. Methods Appl. Sci.*, **18**(4):307–315, 1995, `https://doi.org/10.1002/mma.1670180405`.
10. D. Franco, G. Infante, M.Zima, Second order nonlocal boundary value problems at resonance, *Math. Nachr.*, **284**(7):875–884, 2011, `https://doi.org/10.1002/mana.200810841`.
11. D. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, Boston, MA, 1988, `https://doi.org/10.1016/C2013-0-10750-7`.
12. G. Infante, F. Minhós, P. Pietramala, Non-negative solutions of systems of odes with coupled boundary conditions, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(12):4952–4960, 2012, `https://doi.org/10.1016/j.cnsns.2012.05.025`.

13. G. Infante, M. Zima, Positive solutions of multi-point boundary value problems at resonance, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **69**(8):2458–2465, 2008, `https://doi.org/10.1016/j.na.2007.08.024`.
14. R. Jiang, C. Zhai, Positive solutions for a system of fourth-order differential equations with integral boundary conditions and two parameters, *Nonlinear Anal. Model. Control*, **23**(3):401–422, 2018, `https://doi.org/10.15388/NA.2018.3.7`.
15. A. Leung, A semilinear reaction-diffusion prey-predator system with nonlinear coupled boundary conditions: Equilibrium and stability, *Indiana Univ. Math. J.*, **31**(2):223–241, 1982, `https://doi.org/10.1512/iumj.1982.31.31020`.
16. Z. Lin, C. Xie, On the Dirichlet problem for weakly non-linear elliptic partial differential equations, *Nonlinear Anal., Theory Methods Appl.*, **34**(5):767–778, 1998, `https://doi.org/10.1016/S0362-546X(97)00573-7`.
17. J. Mawhin, *Topological Degree Methods in Nonlinear Boundary-Value Problems*, AMS, Providence, RI, 1979, `https://doi.org/10.1090/cbms/040`.
18. D. O'Regan, M. Zima, Leggett-williams norm-type theorem for coincidences, *Arch. Math.*, **87**(3):233–244, 2006, `https://doi.org/10.1007/s00013-006-1661-6`.
19. W.V. Petryshyn, On the solvability of $x \in Tx + \lambda Fx$ in quasinormal cones with T and F k-set contractive, *Nonlinear Anal., Theory Methods Appl.*, **5**(5):589–591, 1981, `https://doi.org/10.1016/0362-546X(81)90105-X`.
20. J. Santanilla, Some coincidence theorems in wedges, cones, and convex sets, *J. Math. Anal. Appl.*, **105**:357–371, 1985, `https://doi.org/10.1016/0022-247X(85)90053-8`.
21. R. Sun, Z. Bai, Existence of solutions to a system of fractional three-point boundary value problem at resonance, *Appl. Math. Comput.*, **470**:128576, 2024, `https://doi.org/10.1016/j.amc.2024.128576`.
22. J.R.L. Webb, M. Zima, Multiple positive solutions of resonant and non-resonant nonlocal boundary value problems, *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, **71**(3–4):1369–1378, 2009, `https://doi.org/10.1016/j.na.2008.12.010`.
23. S. Xu, G. Zhang, Positive solutions for a second-order nonlinear coupled system with derivative dependence subject to coupled stieltjes integral boundary conditions, *Mediterr. J. Math.*, **19**(2):50, 2022, `https://doi.org/10.1007/s00009-022-01977-9`.
24. C. Yuan, D. Jiang, D. O'Regan, R.P. Agarwal, Multiple positive solutions to systems of nonlinear semipositone fractional diffrential equations with coupled boundary conditions, *Electron. J. Qual. Theory Differ. Equ.*, **13**:1–17, 2012, `https://doi.org/10.14232/ejqtde.2012.1.13`.
25. A. Zettl, *Sturm-Liouville Theory*, AMS, Providence, RI, 2005, `https://doi.org/10.1090/surv/121`.
26. M. Zima, Fixed point theorem of Leggett-Williams type and its application, *J. Math. Anal. Appl.*, **299**(1):254–260, 2004, `https://doi.org/10.1016/j.jmaa.2004.07.002`.