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Asymptotic synchronization for the inertial neural networks by using the differential inequality skills

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Abstract. This paper explores the asymptotic synchronization (AS) for a class of master-slave inertial neural networks (MSINNs). Without employing the existing study approaches such as matrix measure means (MMMs) and linear matrix inequality (LMI), by planning two classes of novel controllers of the trigonometric functions, two criteria to assure the AS for the considered MSINNs are achieved by utilizing the implicit zero point existence theorem of functions and differential inequality way (DIW). By applying the implicit point existence theorem of functions and constructing the controllers of the trigonometric functions, the more concise and more easily verified results on AS can be obtained for neural networks (NNs) than using LMI and MMMs. Our results reveal that the time delay plays a important part in the AS of the considered networks. Namely, when the time delay is less than a certain constant, the AS between the master system and the slave system can be achieved.

Keywords: master-slave inertial neural networks, differential inequality way, asymptotic synchronization, controllers of the trigonometric functions, implicit zero point existence theorem.

1 Introduction

In 1986, Babcock and Westervelt [2] for the first time introduced an inertial term into neural networks (NNs) and pointed out that the dynamical properties could be complicated when the neuron couplings contain an inertial item since the addition of inertial terms can generate very complicated bifurcation behavior and chaos. *In contrast, inertial neural networks* (*INNs*) *provide a more accurate representation of real-world phenomena by incorporating additional terms in the traditional NNs. So, INNs are capable of exhibiting more complex dynamical behaviors compared with traditional NNs*. Since

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then, the dynamical behaviors of all kinds of INNs have been in depth investigated, and some very fruitful results have been achieved [3, 27, 33]. Synchronization for NNs has been deeply studied and used in secure communication, image processing, and so on. Nowadays, the global exponential synchronization (GESN) and AS for the MSINNs have been also deeply studied, and many novel results have been obtained [3,7–9,11,12,16,21–23,25,29,30,32,34]. In [12], the authors were devoted to the GESN for the master-slave impulsive delayed inertial memristor-based quaternion-valued NNs. By using a nonlinear feedback controller, the GESN criteria were established via the Lyapunov stability theory for the studied master-slave NNs. In [32], by using LMI, Lyapunov function, comparison principle, and an extended Halanay differential inequality, the GESN for the master-slave linearly coupled memristive INNs was obtained via the periodically intermittent control. In [22], the MSINNs with mixed delays via quantized pinning controllers were analyzed. In [29], the exponential and adaptive synchronization for a type of master-slave inertial complex-valued NNs were discussed via directly constructing Lyapunov functionals without utilizing standard reduced-order transformation for INNs, and some criteria were derived by using Barbalat lemma and Lyapunov functional. In [25], the synchronization problem of the MSINNs was investigated. By employing LMI approach, the criteria were obtained to assure the synchronization for the considered MSINNs. In [7], the GESN of MSINNs was studied. An extended Halanay differential inequality, matrix decomposition theory, and the average impulsive interval approach were used to derive some conditions to ensure the GESN for the MSINNs. In [30], a class of state-dependent switched INNs with distributed delays were presented. Several new results were derived to ensure the exponential synchronization of such switched NNs by using a novel hybrid control scheme and the Lyapunov stability theory (LSTY). In [34], the AS for the MSINNs was considered. By adopting integral inequality way, some new criteria were derived to ensure the AS for the MSINNs. In [3] and [9], the MMMs were respectively used to study the GESN for their respective discussed networks. In [19], the dissipative synchronization for the inertial memristor-based delayed competitive NNs was studied through an adaptive sliding mode control approach. A novel synchronization criterion addressed in linear matrix inequalities is obtained by adopting the generalized delay-dependent reciprocal convex inequality. Recently, in [5], the AS for a class of master slave complex-valued BAM NNs was discussed. By applying DIW, two criteria to assure the AS of the master-slave NNs were obtained in [5]. In article [14], the synchronization of coupled MSINNs was explored. By utilizing Lyapunov functions, some synchronization criteria were achieved for the MSINNs. In [24], the sampled-data synchronization topic of MSINNs with generally uncertain semi-Markovian jumping was considered. The new extended two-sided looped-function method was applied in the article, and some improved less conservative criteria were achieved to assure the synchronization of the MSINNs. The topic on projective synchronization of delayed inertial quaternion-valued NNs was studied in [11]. By applying the novel Lyapunov functional, several criteria were presented in component form to assure the synchronization of master-slave NNs (MSNNs) in [11]. In [35], the GESN of Clifford-valued memristive fuzzy NNs was studied. To discuss the GESN, a novel and simple linear feedback controller was designed. Sufficient criteria were obtain to ensure the GESN by utilizing the proposed lemma under the designed controller. In [4], the synchronization of semi-Markovian jump competitive MSNNs was studied. By applying an innovative integral inequality technique based on free weighting matrices, sufficient conditions were established for ensuring synchronization. The synchronization criteria are derived in terms of linear matrix inequalities (LMIs).

Up until now, the results on the AS and the GESN for MSNNs and MSINNs have been established mostly by adopting MMMs [3, 9], inequality techniques [7, 16, 21, 26], LSTY [1, 10–12, 14, 16, 18, 20, 24, 28–31], the LMI approach [4, 6, 8, 15, 17, 19, 22, 23, 25, 32], and the integral inequality way [13, 34]. Only in [5] was the new study approach: DIW applied to studying the AS for the MSNNs. In the article, without applying LMI or MMMs, we use the DIW approach introduced in [5] to investigate the AS of the considered MSINNs. *It is a fact that the bounded controllers can better achieve the synchronization than the nonbounded controllers. Nowadays, the authors all apply the nonbounded controllers containing $e_i(t)$, $r_i(t)$ to study the synchronization for MSNNs and MSINNs, where, $e_i(t)$, $r_i(t)$ are two status variables of error system. However, to date, the bounded controllers have not been adopted to study the synchronization for MSNNs and MSINNs.*

Inspired by this point, in our discussion, we adopt bounded controllers of the form $\sin(e_i(t))$ *and* $\cos(e_i(t))$ *to study the synchronization problem for MSINNs.*

In our study, by constructing Lyapunov functional $V(t)$ and applying the error system, we get

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leqslant \sum_{m=1}^{L} \left[-A_m e_m(t) + P_m\bigl(e_m(t)\bigr) - B_m r_m(t) + Q_m\bigl(r_m(t)\bigr)\right],$$

where $e_m(t)$, $r_m(t)$ are two error variables, $P_m(e_m(t))$ is the controller related to $e_m(t)$, $Q(r_m(t))$ is the controller related to $r_m(t)$. Utilizing the implicit zero point existence theorem for continuous functions and DIW, via trigonometric controllers, we obtain the maximum values at the implicit zero points of the considered functions. Finally, let the values of the implicit zero points be smaller than 0. We get

$$-A_m e_m(t) + P_m\bigl(e_m(t)\bigr) \leqslant \max\bigl\{-A_m e_m(t) + P_m\bigl(e_m(t)\bigr)\bigr\} = f(\xi_m) \leqslant 0$$

and

$$-B_m r_m(t) + Q_m\bigl(r_m(t)\bigr) \leqslant \max\bigl\{-B_m r_m(t) + Q_m\bigl(r_m(t)\bigr)\bigr\} = g(\eta_m) \leqslant 0,$$

where ξ_m is the implicit zero point of continuous function $f(x)$, η_m is the implicit zero point of continuous function $g(x)$. This completes the argument for AS.

To date, the results on synchronization obtained for NNs by applying LMI have been all expressed with some very complicated matrix inequalities. Furthermore, these results are very complicated so that they are difficult to verify. Up until now, the researchers generally apply MMMs to study such NNs expressed with a differential equation, though the results obtained and the computation are not complicated. However, when applying MMMs to researching the synchronization for NNs represented with two differential equations, the computation process will be very complicated. So, one of the advantages of our methods is that by using the implicit zero point theorem of functions and constructing the controllers using trigonometric functions, very easily verified synchronization criteria addressed by very concise some algebraic inequality are obtained for the MSINNs in our study. The other of the advantages of our method is that our method still can be used to study the finite-time synchronization for MSNNs.

In our study on the AS of the MSINNs, the *main difficulties and challenges lie in how to design the trigonometric controllers to realize AS and how to construct suitable functions that satisfy the implicit zero point existence theorem in order to achieve AS for MSINNs.* Since the implicit zero point existence theorem and trigonometric controllers have rarely been used to study AS for MSNNs, our study is of definite significance. So, the main contributions of the article are as follows:

(i) In the proof of the AS, the implicit zero existence theorem of functions is used to investigate the AS for the MSINNs.

(ii) Second, the controllers based on trigonometric functions are designed to study the AS for the MSINNs, which represents novel contribution.

(iii) Finally, two novel criteria are established to achieve the AS for the considered MSINNs by using the implicit zero existence theorem of functions and designing the trigonometric controllers.

2 Preliminaries

Introducing an inertial term $\mathrm{d}^2x_m(t)/\mathrm{d}t^2$ into NNs

$$0 = -c_m\frac{\mathrm{d}x_m(t)}{\mathrm{d}t} - d_m x_m(t) + G_m\big(x(t)\big),$$

where

$$G_m\big(x(t)\big) = \sum_{n=1}^{L} a_{mn} f_n\big(x_n(t)\big) + \sum_{n=1}^{L} b_{mn} f_n\big(x_n\big(t-\tau_{mn}(t)\big)\big) + I_m,$$

we get the INNs with time-varying delays depicted by

$$\frac{\mathrm{d}^2x_m(t)}{\mathrm{d}t^2} = -c_m\frac{\mathrm{d}x_m(t)}{\mathrm{d}t} - d_m x_m(t) + G_m\big(x(t)\big) \tag{1}$$

where $x(t) = (x_1(t), x_2(t), \dots, x_l(t))^{\mathrm{T}} \in \mathbb{R}^l$; $x_m(t)$ denotes the state variables of mth neuron at time t, $m = 1, 2, \dots, l$. The second derivative is called an inertial term of system (1), $c_m, d_m > 0$, $\tau_{mn}(t) > 0$ is the time delay function, $\max\{\tau_{mn}(t), \tau'_{mn}(t)\} \leqslant \hat{\tau} < 1$; I_m denotes the inputs, $f_n(\cdot)$ is the neuron activation function.

The initial values (IVs) of system (1) are given as follows:

$$x_m(s) = \phi_{x_m}(s), \qquad \frac{\mathrm{d}x_m(s)}{\mathrm{d}s} = \psi_{x_m}(s), \quad s \in [-\hat{\tau}, 0].$$

Tn this study, for the function $f_n(\cdot)$ in system (1), we always assume that

($\widehat{\mathrm{H}}_1$) There exist constants $l_n > 0$ such that $|f_n(\hat{x}) - f_n(\hat{y})| \leqslant l_n|\hat{x} - \hat{y}|$ for all $\hat{x}, \hat{y} \in \mathbb{R}$, $n = 1, 2, \dots, l$; $|\cdot|$ is the norm of the Euclidean space $\mathbb{R}$.

Let

$$v_m(t) = k_m \frac{\mathrm{d}x_m(t)}{\mathrm{d}t} + \lambda_m x_m(t), \quad m = 1, 2, \dots, l,$$

where $k_m \neq 0$, λ_m are two chosen constants. Then system (1) can be rewritten as

$$\begin{aligned} \frac{\mathrm{d}x_m(t)}{\mathrm{d}t} &= -\frac{\lambda_m}{k_m} x_m(t) + \frac{v_m(t)}{k_m}, \\ \frac{dv_m(t)}{\mathrm{d}t} &= -\left[c_m - \frac{\lambda_m}{k_m}\right] v_m(t) + \left[c_m \lambda_m - k_m d_m - \frac{\lambda_m^2}{k_m}\right] x_m(t) \\ &\quad + k_m G_m\big(x(t)\big). \end{aligned} \tag{2}$$

The IVs of system (2) can be written as

$$x_m(s) = \phi_{xm}(s), \quad v_m(s) = k_m \psi_{xm}(s) + \lambda_m \phi_{xm}(s), \quad s \in [-\hat{\tau}, 0],$$

where $\phi_{xm}(s)$ and $\psi_{xm}(s)$ are bounded continuous functions.

We refer to (2) as the master system and consider the slave INNs as follows:

$$\begin{aligned} \frac{\mathrm{d}y_m(t)}{\mathrm{d}t} &= -\frac{\lambda_m}{k_m} y_m(t) + \frac{u_m(t)}{k_m} + \widehat{P}_m(t), \\ \frac{\mathrm{d}u_m(t)}{\mathrm{d}t} &= -\left[c_m - \frac{\lambda_m}{k_m}\right] u_m(t) + \left[c_m \lambda_m - k_m d_m - \frac{\lambda_m^2}{k_m}\right] y_m(t) \\ &\quad + k_m G_m\big(y(t)\big) + \widehat{Q}_m(t), \end{aligned} \tag{3}$$

where the parameters are the same as those in system (2), $\widehat{P}_m(t)$, $\widehat{Q}_n(t)$ are the controllers.

The IVs of system (3) are given as follows:

$$y_m(s) = \phi_{ym}(s) \quad u_m(s) = k_m \psi_{ym}(s) + \lambda_m \phi_{ym}(s), \quad s \in [-\hat{\tau}, 0],$$

where $\phi_{ym}(s)$ and $\psi_{ym}(s)$ are bounded continuous functions.

Definition 1. The master system (2) and the slave system (3) can achieve AS if for arbitrary solutions of systems (2) and (3) denoted by $[x_1(t), x_2(t), \dots, x_l(t), v_1(t), v_2(t), \dots, v_l(t)]^{\mathrm{T}}$ and $[y_1(t), y_2(t), \dots, y_l(t), u_1(t), u_2(t), \dots, u_l(t)]^{\mathrm{T}}$, we have

$$\lim_{t\to\infty} \big|y_m(t) - x_m(t)\big| = 0, \qquad \lim_{t\to\infty} \big|u_m(t) - v_m(t)\big| = 0.$$

Definition 2. If equation $f(x) = 0$ has a root x_0, but this root cannot be solved explicitly, then, by applying the zero point existence theorem for continuous functions, we can prove that $f(x)$ has a zero point x_0. Such a point x_0 is called an implicit zero point of the function $f(x)$.

3 Main results

Let $e_m(t) = y_m(t) - x_m(t)$ and $r_m(t) = u_m(t) - v_m(t)$. Then the error INNs can be described as follows:

$$\begin{aligned}\frac{de_m(t)}{\mathrm{d}t} &= -\frac{\lambda_m}{k_m}e_m(t) + \frac{r_m(t)}{k_m} + \widehat{P}_m(t),\\ \frac{dr_m(t)}{\mathrm{d}t} &= -\left[c_m - \frac{\lambda_m}{k_m}\right]r_m(t) + \left[c_m\lambda_m - k_m d_m - \frac{\lambda_m^2}{k_m}\right]e_m(t)\\ &\quad + k_m\left[G_m\big(y(t)\big) - G_m\big(x(t)\big)\right] + \widehat{Q}_m(t).\end{aligned} \tag{4}$$

The controllers in system (4) are designed for $m = 1, 2, \dots, l$:

$$\begin{aligned}\widehat{P}_m(t) &= -\operatorname{sign}\left[e_m(t)\right]\left(1 + \left[e_m(t)\right]^2 + \exp\big(e_m(t)\big)\right),\\ \widehat{Q}_m(t) &= r_m(t)\left(\beta_1\cos\big(2\left|r_m(t)\right|\big) + \beta_2\sin\big(\left|r_m(t)\right|\big) + \beta_3\right)\end{aligned} \tag{5}$$

and

$$\begin{aligned}\widehat{P}_m(t) &= e_m(t)\left(\beta_5 + \frac{\alpha_2\cos(|e_m(t)|)}{\sin(|e_m(t)|) - 2}\right),\\ \widehat{Q}_m(t) &= \operatorname{sign}\left[r_m(t)\right]\left(\beta_4 - r_m^2(t) - exp\big(\left|r_m(t)\right|\big)\right),\end{aligned} \tag{6}$$

where $\beta_1 < 0$, $\beta_2 > 0$, $\beta_4 < -1$, β_3, β_5, α_2 are constants.

The following notations are introduced:

$$\begin{aligned}A_m &= -\frac{\lambda_m}{k_m} + c_m|\lambda_m| + |k_m|d_m + \frac{\lambda_m^2}{|k_m|} + \sum_{n=1}^{L} l_m|a_{nm}||k_n| + \frac{\sum_{n=1}^{L} l_m|k_n||b_{nm}|}{1-\hat{\tau}},\\ B_m &= -\left[c_m - \beta_3 - \frac{\lambda_m}{k_m}\right] + \frac{1}{|k_m|},\\ C_m &= \left(\beta_5 - \frac{\lambda_m}{k_m}\right) + c_m|\lambda_m| + |k_m|d_m + \frac{\lambda_m^2}{|k_m|} + \sum_{n=1}^{L} l_m|a_{nm}||k_n| + \frac{\sum_{n=1}^{L} l_m|k_n||b_{nm}|}{1-\hat{\tau}}.\end{aligned}$$

Theorem 1. *Suppose that* ($\widehat{\mathrm{H}}_1$) *holds. Then systems* (2) *and* (3) *can realize AS via controllers* (5) *if the following conditions hold*:

($\widehat{\mathrm{H}}_2$) $A_m < 4$, $m = 1, 2, \dots, l$;
($\widehat{\mathrm{H}}_3$) $B_m - \beta_1 + \beta_2 < 0$, $m = 1, \dots, l$.

Proof. Set

$$\widehat{U}(t) = \widehat{U}_1(t) + \widehat{U}_2(t),$$

where

$$\widehat{U}_1(t) = \sum_{m=1}^{L}\left(\left|e_m(t)\right| + \left|r_m(t)\right|\right),$$

$$\widehat{U}_2(t) = \frac{1}{1-\hat{\tau}} \sum_{m=1}^{L} \sum_{n=1}^{L} |k_m| l_n |b_{mn}| \int\limits_{t-\tau_{mn}(t)}^{t} \big|e_n(s)\big| \, \mathrm{d}s.$$

From the error system (4) one has

$$\begin{aligned}
\widehat{U}_1'(t) = {} & \sum_{m=1}^{L} \biggl\{ \operatorname{sign}\bigl[e_m(t)\bigr] \biggl(-\frac{\lambda_m}{k_m} e_m(t) + \frac{1}{k_m} r_m(t) + \widehat{P}_m(t) \biggr) \\
& + \operatorname{sign}\bigl[r_m(t)\bigr] \biggl(-\biggl[c_m - \frac{\lambda_m}{k_m}\biggr] r_m(t) + \biggl[c_m \lambda_m - k_m d_m - \frac{\lambda_m^2}{k_m}\biggr] e_m(t) \\
& + k_m \bigl[G_m\bigl(y(t)\bigr) - G_m\bigl(u_m(t)\bigr)\bigr] + \widehat{Q}_m(t) \biggr) \biggr\} \\
\leqslant {} & \sum_{m=1}^{L} \biggl\{ -\frac{\lambda_m}{k_m} |e_m(t)| + \frac{1}{|k_m|} \big|r_m(t)\big| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\
& - \biggl[c_m - \beta_3 - \frac{\lambda_m}{k_m}\biggr] \big|r_m(t)\big| + \biggl[c_m |\lambda_m| + |k_m| d_m + \frac{\lambda_m^2}{|k_m|}\biggr] \big|e_m(t)\big| \\
& + |k_m| \big|G_m\bigl(y(t)\bigr) - G_m\bigl(x(t)\bigr)\big| \\
& + \big|r_m(t)\big| \bigl[\beta_1 \cos\bigl(2\big|r_m(t)\big|\bigr) + \beta_2 \sin\bigl(\big|r_m(t)\big|\bigr)\bigr] \biggr\} \\
\leqslant {} & \sum_{m=1}^{l} \biggl\{ -\frac{\lambda_m}{k_m} \big|e_m(t)\big| + \frac{1}{|k_m|} \big|r_m(t)\big| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\
& - \biggl[c_m - \beta_3 - \frac{\lambda_m}{k_m}\biggr] \big|r_m(t)\big| + \biggl[c_m |\lambda_m| + |k_m| d_m + \frac{\lambda_m^2}{|k_m|}\biggr] \big|e_m(t)\big| \\
& + |k_m| \sum_{n=1}^{L} l_n \bigl[|a_{mn}| \big|e_n(t)\big| + |b_{mn}| \big|e_n\bigl(t - \tau_{mn}(t)\bigr)\big|\bigr] \\
& + \big|r_m(t)\big| \bigl[\beta_1 \cos\bigl(2\big|r_m(t)\big|\bigr) + \beta_2 \sin\bigl(\big|r_m(t)\big|\bigr)\bigr] \biggr\} \\
\leqslant {} & \sum_{m=1}^{L} \biggl\{ -\frac{\lambda_m}{k_m} \big|e_m(t)\big| + \frac{1}{|k_m|} \big|r_m(t)\big| - \biggl[c_m - \beta_3 - \frac{\lambda_m}{k_m}\biggr] \big|r_m(t)\big| \\
& + \biggl[c_m |\lambda_m| + |k_m| |d_m| + \frac{\lambda_m^2}{|k_m|}\biggr] \big|e_m(t)\big| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\
& + |k_m| \sum_{n=1}^{L} l_n \bigl[|a_{mn}| \big|e_n(t)\big| + \big|e_n\bigl(t - \tau_{mn}(t)\bigr)\big| |b_{mn}|\bigr] \\
& + \big|r_m(t)\big| \bigl[\beta_1 \cos\bigl(2\big|r_m(t)\big|\bigr) + \beta_2 \sin\bigl(\big|r_m(t)\big|\bigr)\bigr] \biggr\}. \qquad (7)
\end{aligned}$$

Furthermore, one has

$$\begin{aligned}\widehat{U}_2'(t) &= \frac{1}{1-\hat{\tau}} \sum_{m=1}^{L} \sum_{n=1}^{L} |k_m| l_n |b_{mn}| \bigl(\bigl|e_n(t)\bigr| - \bigl(1-\tau_{mn}'(t)\bigr)\bigl|e_n\bigl(t-\tau_{mn}(t)\bigr)\bigr|\bigr) \\ &\leqslant \frac{1}{1-\hat{\tau}} \sum_{m=1}^{L} \sum_{n=1}^{L} |k_m| l_n |b_{mn}| \bigl|e_n(t)\bigr| \\ &\quad - \sum_{m=1}^{L} \sum_{n=1}^{L} |k_m| l_n |b_{mn}| \bigl|e_n\bigl(t-\tau_{mn}(t)\bigr)\bigr|. \end{aligned} \tag{8}$$

From (7) and (8) one has

$$\begin{aligned}\widehat{U}'(t) &\leqslant \sum_{m=1}^{L} \biggl\{ -\frac{\lambda_m}{k_m}\bigl|e_m(t)\bigr| + \frac{1}{|k_m|}\bigl|r_m(t)\bigr| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\ &\quad - \biggl[c_m - \beta_3 - \frac{\lambda_m}{k_m}\biggr]\bigl|r_m(t)\bigr| + \biggl[c_m|\lambda_m| + |k_m| d_m + \frac{\lambda_m^2}{|k_m|}\biggr]\bigl|e_m(t)\bigr| \\ &\quad + |k_m| \sum_{n=1}^{L} l_n \biggl[|a_{mn}|\bigl|e_n(t)\bigr| + \frac{1}{1-\hat{\hat{\tau}}}|b_{mn}|\bigl|e_n(t)\bigr|\biggr] \\ &\quad + \bigl|r_m(t)\bigr|\bigl[\beta_1 \cos\bigl(2\bigl|r_m(t)\bigr|\bigr) + \beta_2 \sin\bigl(\bigl|r_m(t)\bigr|\bigr)\bigr] \biggr\} \\ &= \sum_{m=1}^{L} \bigl\{ A_m\bigl|e_m(t)\bigr| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\ &\quad + \bigl|r_m(t)\bigr|\bigl[B_m + \beta_1 \cos\bigl(2\bigl|r_m(t)\bigr|\bigr) + \sin\bigl(\bigl|r_m(t)\bigr|\bigr)\beta_2\bigr]\bigr\} \\ &< \sum_{m=1}^{L} \bigl\{ 4\bigl|e_m(t)\bigr| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) \\ &\quad + \bigl|r_m(t)\bigr|\bigl[B_m + \beta_1 \cos\bigl(2\bigl|r_m(t)\bigr|\bigr) + \beta_2 \sin\bigl(\bigl|r_m(t)\bigr|\bigr)\bigr]\bigr\}. \end{aligned} \tag{9}$$

Let $F(x) = 4x - 1 - x^2 - \exp(x)$, $x \geqslant 0$. Then $F'(x) = 4 - 2x - \exp(x)$ and $F''(x) = -\exp(x) - 2$. Thus, $F'(x)$ is decreasing in x. Moreover, $F'(0) = 3 > 0$ and $F'(1) = 2 - \mathrm{e} < 0$. Hence, there exists a point $x_0 \in (0, 1)$ such that $F'(x_0) = 0$, that is, $\exp(x_0) + 2x_0 = 4$. Because $F'(x) > 4 - 2x_0 - \exp(x_0) = 0$ for $x < x_0$ and $F'(x) < 4 - 2x_0 - \exp(x_0) = 0$ for $x > x_0$, it follows that

$$\begin{aligned}\max F(x) = F(x_0) &= 4x_0 - 1 - x_0^2 - \exp(x_0) = 2x_0 - 4 + 4x_0 - 1 - x_0^2 \\ &= -\bigl(x_0^2 - 6x_0 + 5\bigr) = -(x_0 - 1)(x_0 - 5) < 0.\end{aligned}$$

Let $x = |e_m(t)|$. Then

$$4\bigl|e_m(t)\bigr| - 1 - \bigl[e_m(t)\bigr]^2 - \exp\bigl(e_m(t)\bigr) < 0. \tag{10}$$

Let $G(\sin y) = B_m + \beta_1 \cos 2y + \beta_2 \sin y$, $y \geqslant 0$. Then

$$\begin{aligned} G(\sin y) &= B_m + \beta_1\bigl(1 - 2\sin^2 y\bigr) + \beta_2 \sin y \\ &= B_m + \beta_1 - 2\beta_1\left(\sin^2 y - \frac{\beta_2}{2\beta_1}\sin y\right) \\ &= B_m + \beta_1 - 2\beta_1\left(\sin y - \frac{\beta_2}{4\beta_1}\right)^2 + \frac{\beta_2^2}{8\beta_1}. \end{aligned} \tag{11}$$

There are two possible cases to consider:

(i) $\beta_2/(4\beta_1) \in (-1, 1)$;
(ii) $\beta_2/(4\beta_1) \in (-\infty, -1) \cup (1, \infty)$.

(i) When $\beta_2/(4\beta_1) \in (-1, 1)$, we have from (11)

$$\begin{aligned} G(1) &= B_m + \beta_1 - 2\beta_1\left(1 - \frac{\beta_2}{4\beta_1}\right)^2 + \frac{\beta_2^2}{8\beta_1} = B_m - \beta_1 + \beta_2, \\ G(-1) &= B_m + \beta_1 - 2\beta_1\left(1 + \frac{\beta_2}{4\beta_1}\right)^2 + \frac{\beta_2^2}{8\beta_1} = B_m - \beta_1 - \beta_2, \\ G\left(\frac{\beta_2}{4\beta_1}\right) &= B_m + \beta_1 + \frac{\beta_2^2}{8\beta_1}. \end{aligned}$$

In view of $\beta_1 < 0$, $\beta_2 > 0$, $\max\bigl\{G(\sin y)\bigr\} = B_m - \beta_1 + \beta_2$.

(ii) When $\beta_2/(4\beta_1) \in (-\infty, -1) \cup (1, \infty)$, we get

$$\begin{aligned} G(1) &= B_m + \beta_1 - 2\beta_1\left(1 - \frac{\beta_2}{4\beta_1}\right)^2 + \frac{\beta_2^2}{8\beta_1} = B_m - \beta_1 + \beta_2, \\ G(-1) &= B_m + \beta_1 - 2\beta_1\left(1 + \frac{\beta_2}{4\beta_1}\right)^2 + \frac{\beta_2^2}{8\beta_1} = B_m - \beta_1 - \beta_2. \end{aligned}$$

Then $\max\{G(\sin y)\} = B_m - \beta_1 + \beta_2$.

Based on (i) and (ii), we obtain

$$\max\bigl\{G(\sin y)\bigr\} = B_m - \beta_1 + \beta_2.$$

By ($\widehat{\mathrm{H}}_3$),

$$B_m + \beta_1 \cos(2y) + \beta_2 \sin y \leqslant B_m - \beta_1 + \beta_2 < 0, \quad y \geqslant 0.$$

Let $y = |r_m(t)|$, then

$$B_m + \beta_1 \cos(2y) + \beta_2 \sin y < 0. \tag{12}$$

Substituting (10) and (12) into (9), it follows

$$U'(t) \leqslant 0.$$

This completes the proof of Theorem 1. □

Theorem 2. *Suppose that* $(\widehat{\mathrm{H}}_1)$ *holds. Then systems* (2) *and* (3) *can achieve AS via controllers* (6) *if the following conditions hold*:

$(\widehat{\mathrm{H}}_4)$ $C_m + |\alpha_2|/\sqrt{3} < 0$, $m = 1, \dots, l$;
$(\widehat{\mathrm{H}}_5)$ $1 < B_m < 2 + \mathrm{e}$, $m = 1, \dots, l$.

Proof. We construct $\widehat{U}(t)$ as in Theorem 1. From system (3) we obtain

$$\begin{aligned}
\widehat{U}'(t) \leqslant \sum_{m=1}^{L} \Biggl\{ & \biggl(\beta_5 - \frac{\lambda_m}{k_m}\biggr)\bigl|e_m(t)\bigr| + \frac{1}{|k_m|}\bigl|r_m(t)\bigr| \\
& - \biggl[c_m - \frac{\lambda_m}{k_m}\biggr]\bigl|r_m(t)\bigr| + \biggl[c_m|\lambda_m| + |k_m|d_m + \frac{\lambda_m^2}{|k_m|}\biggr]\bigl|e_m(t)\bigr| \\
& + |k_m| \sum_{n=1}^{L} l_n \biggl[|a_{mn}|\bigl|e_m(t)\bigr| + \frac{1}{1-\hat{\hat{\tau}}}|b_{mn}|\bigl|e_n(t)\bigr|\biggr] \\
& + \bigl|e_m(t)\bigr| \frac{\alpha_2 \cos(|e_m(t)|)}{\sin(|e_m(t)|) - 2} + \beta_4 - \bigl|r_m(t)\bigr|^2 - \exp\bigl(\bigl|r_m(t)\bigr|\bigr) \Biggr\} \\
= \sum_{m=1}^{L} \Biggl\{ & \bigl|e_m(t)\bigr| \biggl[C_m + \frac{\alpha_2 \cos(|e_m(t)|)}{\sin(|e_m(t)|) - 2}\biggr] + B_m\bigl|r_m(t)\bigr| + \beta_4 \\
& - \bigl|r_m(t)\bigr|^2 - \exp\bigl(\bigl|r_m(t)\bigr|\bigr) \Biggr\}.
\end{aligned} \tag{13}$$

Let $z = \alpha_2 \cos(|e_m(t)|)/(\sin(|e_m(t)|) - 2)$. Then

$$z \sin\bigl(\bigl|e_m(t)\bigr|\bigr) - 2z = \alpha_2 \cos\bigl(\bigl|e_m(t)\bigr|\bigr).$$

Hence,

$$\begin{aligned}
2z &= z \sin\bigl(\bigl|e_m(t)\bigr|\bigr) - \alpha_2 \cos\bigl(\bigl|e_m(t)\bigr|\bigr) \\
&= \sqrt{z^2 + \alpha_2^2} \sin\bigl(\bigl|e_m(t)\bigr| + \theta\bigr) \leqslant \sqrt{z^2 + \alpha_2^2},
\end{aligned} \tag{14}$$

where θ satisfies $\sin\theta = -\alpha_2/\sqrt{z^2 + \alpha_2^2}$ and $\cos\theta = z/\sqrt{z^2 + \alpha_2^2}$. By (14), it follows that $(2z)^2 \leqslant z^2 + \alpha_2^2$, that is, $3z^2 - \alpha_2^2 \leqslant 0$. So $|z| \leqslant |\alpha_2|/\sqrt{3}$, which, by $(\widehat{\mathrm{H}}_4)$, results in

$$C_m + \frac{\alpha_2 \cos(|e_m(t)|)}{\sin(|e_m(t)|) - 2} \leqslant C_m + \frac{|\alpha_2|}{\sqrt{3}} < 0. \tag{15}$$

Let $h(x) = B_m x + \beta_4 - x^2 - \exp(x)$, $x \geqslant 0$. Then $h'(x) = B_m - 2x - \exp(x)$. It is clear that $h'(x)$ is decreasing in x. Because $h'(0) = B_m - 1 > 0$, $h'(1) = B_m - 2 - e < 0$, according to the existence theorem of zero point, there is a point $x_0 \in (0, 1)$ such that $h'(x_0) = 0$. That is, $B_i - 2x_0 - \exp(x_0) = 0$. Then we have $h'(x) > 0$ when $x \in (-\infty, x_0)$, and $h'(x) < 0$ when $x \in (x_0, +\infty)$. Hence,

$$\begin{aligned}
\max\bigl\{h(x)\bigr\} &= h(x_0) = B_m x_0 + \beta_4 - x_0^2 - \exp(x_0) \\
&= (B_m + 2)x_0 - x_0^2 + \beta_4 - B_m, \quad 0 < x_0 < 1.
\end{aligned} \tag{16}$$

Since $(4+\mathrm{e})/2 > 1$, $h(x_0 = 0) = \beta_4 - B_m < \beta_4$, $h(x_0 = 1) = 1 + \beta_4$. By (16), one has

$$\max_{0<x_0<1}\{h(x_0)\} < 1 + \beta_4. \tag{17}$$

By (16) and (17), we have

$$h(x) < 1 + \beta_4. \tag{18}$$

Let $x = |r_m(t)|$. Since $\beta_4 < -1$, from (18) one has

$$\begin{aligned} &B_m|r_m(t)| + \beta_4 - |r_m(t)|^2 - \exp(|r_m(t)|) \\ &\quad \leqslant 1 + \beta_4 + 0.25B_m^2 \leqslant 1 + \beta_4 \leqslant 0. \end{aligned} \tag{19}$$

Substituting (15) and (19) into (13), we have

$$\widehat{U}'(t) < 0.$$

This completes the proof of Theorem 2. □

Remark 1. In many papers [1, 6–10, 12, 13, 15–17, 20–23, 25, 26, 28–32, 34] investigating AS for the MSNNs, synchronization criteria were attained chiefly by adopting LSTY [1, 10–12, 14, 16, 20, 24, 28–31], inequality techniques [7, 16, 21, 26], LMI way [4, 6, 8, 15, 17, 19, 22, 23, 25, 32], and MMMs [3, 9]. However, in our work, the implicit zero point existence theorem, DIW, and two classes of trigonometric controllers are used to achieve AS for the considered MSINNs, instead of using the aforementioned methods. Although DIW is also used in [5] to study AS, the specific inequality techniques in our work are quite different from those in [5].

Remark 2. Two classes of trigonometric controllers in our study are quite different from those in the previous papers [1, 4–10, 12, 13, 15–17, 19–23, 25, 26, 28–32, 34]. In our discussion, the constructed controllers contain the terms $\sin(|e_m(t)|)$, $\cos(|e_m(t)|)$, $\sin(|r_m(t)|)$, and $\cos(|r_m(t)|)$, while in past literature, the controllers only contain $|e_m(t)|$, $|r_m(t)|$, where $r_m(t)$, $e_m(t)$ are the status variables of the error system. The inequality skills are also different from those in the previous articles [1, 4–10, 12, 13, 15–17, 19–23, 25, 26, 28–32, 34]. In our discussion, inequalities of $\sin(e_i(t))$ and $\cos(r_i(t))$ are used to study synchronization, while in the past literature, only the inequalities of $e_i(t)$, $r_i(t)$ were used to study synchronization, where, $e_i(t)$, $r_i(t)$ are status variables of the error system.

Remark 3. The results on synchronization obtained by adopting LMI and MMMs are so complicated that they are difficult to verify. However, because our results are addressed in the simple algebraic inequalities, thus our results are more concise and more easily checked and tested.

Remark 4. Equations of elastic force in mechanics, such as $x'' + \delta x' + \alpha x + f(x)$ and Duffing equations all are the special cases of INNs. So the study on INNs has also practical meaning.

4 Numerical examples

In order to represent our results, we simulate two examples in this section.

Example 1. We consider the master system (2), the slave system (3), and the error system (4) with controllers (5) for $m = 1, 2$, $n = 1, 2$, where

$$(a_{mn}) = \begin{pmatrix} 0.1 & 0.4 \\ 0.5 & 0.3 \end{pmatrix}, \qquad (b_{mn}) = \begin{pmatrix} 0.3 & 1 \\ 0.2 & 0.5 \end{pmatrix},$$
$$(h_{mn}) = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.1 \end{pmatrix},$$

$c_1 = 20 > 0$, $c_2 = 15 > 0$, $d_1 = 5 > 0$, $d_2 = 12 > 0$, $\alpha_1 = 0.2 < 0.5$, $\beta_1 = -0.5 < 0$, $\beta_2 = 0.5 > 0$, $\beta_3 = -3$, $\lambda_1 = -5$, $\lambda_2 = -3$, $k_m = l_j = 1$, $I_1 = 2$, $I_2 = 1$, $f_n(x_n(t)) = |x_n(t)| + 0.5$, $\tau_{mn}(t) = 0.1\cos(t) + 0.2 > 0$, $\tau'_{mn}(t) = -0.1\sin(t) \leqslant \hat{\tau} = 0.5$. Therefore,

$$A_1 = 2 < 4, \qquad B_1 - \beta_1 + \beta_2 = -24.8 < 0,$$
$$A_2 = 1 < 4, \qquad B_2 - \beta_1 + \beta_2 = -18 < 0.$$

Thus $(\widehat{\mathrm{H}}_1)$–$(\widehat{\mathrm{H}}_3)$ in Theorem 1 are satisfied. Based on Theorem 1, systems (2) and (3) achieve AS via controller (5). The approach, the designed controllers, and the results on AS for systems (2) and (3) in this example are different from those in the literature such as [3,5,9] using MMMs, [7,16,21,26] using inequality techniques, [1,10,12,16,20,28–31] using LSTY, [6, 8, 15, 17, 22, 23, 25, 32] using LMI, and [13, 34] using integral inequality method. Moreover, it is not difficult to verify that the parameters do not satisfy the AS conditions in the above papers. Therefore, the AS of systems (2) and (3) cannot be verified with the previous results in above articles.

The curves of drive inertial neural networks with time-varying delays are shown in the following Fig. 1, and the curves of corresponding response systems are shown in the following Fig. 2, the phase difference and amplitude fluctuation of the drive and response system are obvious in the time domain waveform. The error system's curves are shown in the following Fig. 3.

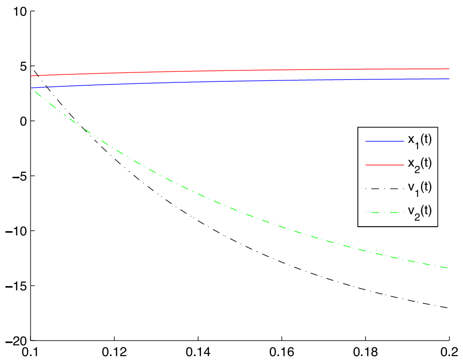

Figure 1. Curves of the drive system in Example 1.

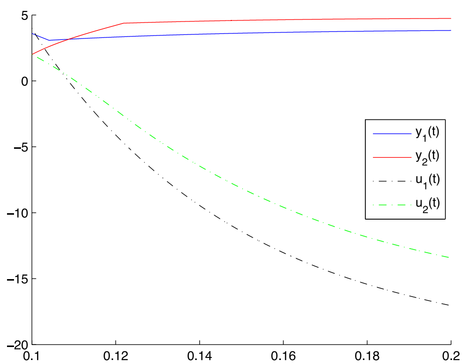

Figure 2. Curves of the response system in Example 1.

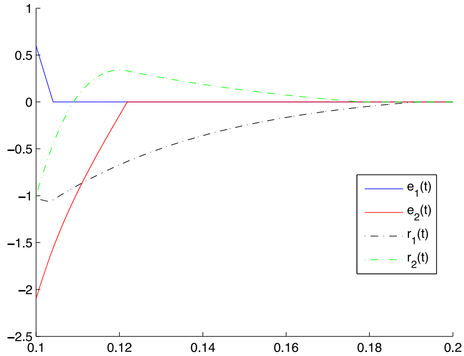

Figure 3. Curves of the error system in Example 1.

Example 2. We consider the following INNs:

$$\frac{\mathrm{d}^2 x_m(t)}{\mathrm{d}t^2} = -c_m \frac{\mathrm{d}x_m(t)}{\mathrm{d}t} - d_m x_m(t) + G_m\bigl(x(t)\bigr). \tag{20}$$

System (20) is, in physical terms, a Duffing oscillator equation with elasticity, where $c_m > 0$ is the damping constant, and $d_m x_m(t) + G_m(x_m(t))$ denotes the elastic restoring force. Hence, the study of synchronization for MSINNs (20) plays an important role due to its practical applications. We consider the oscillator equation with elasticity (2), system (3), and the error system (4) under controllers (6) for $m = 1, 2, 3$, $n = 1, 2, 3$, where

$$(a_{mn}) = \begin{pmatrix} 0.3 & 0.2 & 0.1 \\ 0.4 & 0.2 & 0.3 \\ 0.2 & 0.1 & 0.4 \end{pmatrix}, \qquad (b_{mn}) = \begin{pmatrix} 0.1 & 0.5 & 0.2 \\ 0.3 & 0.2 & 0.4 \\ 0.1 & 0.3 & 0.2 \end{pmatrix},$$

$$(h_{mn}) = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.1 & 0.5 & 0.2 \\ 0.3 & 0.1 & 0.5 \end{pmatrix},$$

$c_1 = 0.2 > 0$, $c_2 = 0.1 > 0$, $c_3 = 0.3 > 0$, $d_1 = 0.1 > 0$, $d_2 = 0.3 > 0$, $d_3 = 0.2 > 0$, $\alpha_2 = 1$, $\beta_3 = 3$, $\beta_4 = -1.5 < -1$, $\beta_5 = -10$, $\lambda_1 = 1$, $\lambda_2 = 2$, $\lambda_3 = 0.5$, $k_1 = 4$, $k_2 = 5$, $k_3 = 3$, $l_m = 0.5$; $I_1(t) = t + 1$, $I_2(t) = \sin(t) - 1$, $I_3(t) = \cos(t)$, $f_n(x_n(t)) = 0.5|x_n(t) + 1|$, $\tau_{ij}(t) = 0.2\cos(t) + 0.25 > 0$, $\tau'_{ij}(t) = -0.2\sin(t) \leqslant \hat{\tau} = 0.5$. The initial conditions are defined as follows: $x_1(0) = 10$, $x_2(0) = 6$, $x_3(0) = 3$, $v_1(0) = 2$, $v_2(0) = 3$, $v_3(0) = 5$, $y_1(0) = 5$, $y_2(0) = 1$, $y_3(0) = 4$, $u_1(0) = 4$, $u_2(0) = 2$, $u_3(0) = 1$. Therefore, we have $(\widehat{\mathrm{H}}_1)$ and $(\widehat{\mathrm{H}}_4)$:

$$C_1 + \frac{|\alpha_2|}{\sqrt{3}} = -4.1726 < 0, \qquad C_2 + \frac{|\alpha_2|}{\sqrt{3}} = -0.6126 < 0,$$

$$C_3 + \frac{|\alpha_2|}{\sqrt{3}} = -3.1893 < 0;$$

and $(\widehat{\mathrm{H}}_5)$:

$$1 < B_1 = 3.85 < 2 + \mathrm{e} = 4.7183, \qquad 1 < B_2 = 4.38 < 2 + \mathrm{e} = 4.7183,$$
$$1 < B_3 = 4.4333 < 2 + \mathrm{e} = 4.7183.$$

Therefore, $(\widehat{\mathrm{H}}_1)$, $(\widehat{\mathrm{H}}_4)$, and $(\widehat{\mathrm{H}}_5)$ hold. Then, based on Theorem 2, systems (2) and (3) can achieve AS via controllers (6). Since the approach, the designed controllers, and the obtained results differ from those in most papers on AS, the AS of systems (2) and (3) cannot be verified by the results in [1, 5–10, 12, 13, 15–17, 20–23, 25, 26, 28–32, 34].

The driver's 3D trajectories $x_1(t)$, $x_2(t)$, $x_3(t)$ in system (20) are shown in Fig. 4, while the responder's trajectories $y_1(t)$, $y_2(t)$, $y_3(t)$ are presented in Fig. 5. These plots illustrate the state evolution of an inertial system controlled by a neural network. The 2D phase diagram shows periodic oscillations of the master system after variable substitution. The slave INNs and corresponding response system curves are given in Figs. 6 and 7.

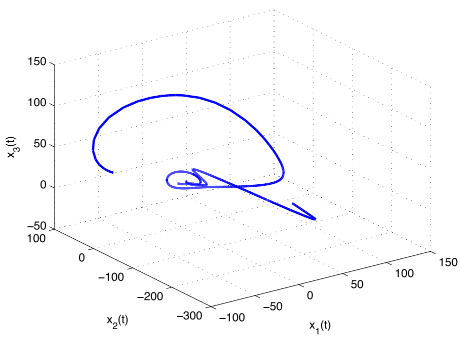

Figure 4. The 3D image of the drive system in the Example 2.

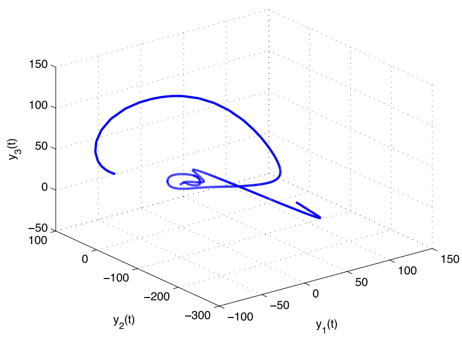

Figure 5. The 3D image of the response system in Example 2.

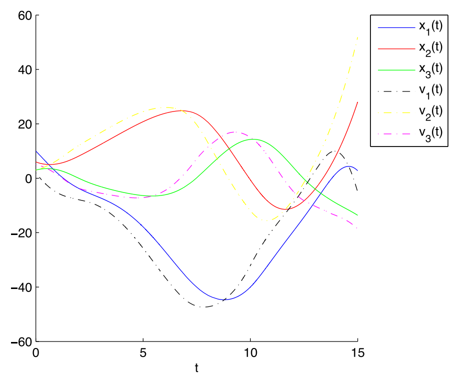

Figure 6. Curves of the drive system after variable transform in the Example 2.

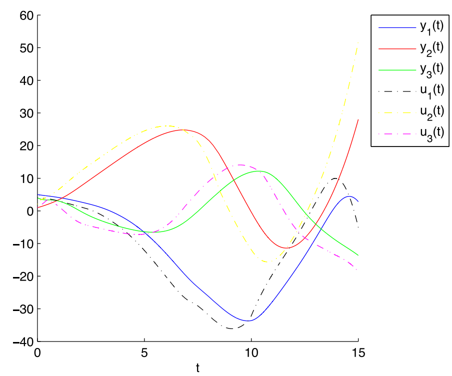

Figure 7. Curves of the response system after variable transform in Example 2.

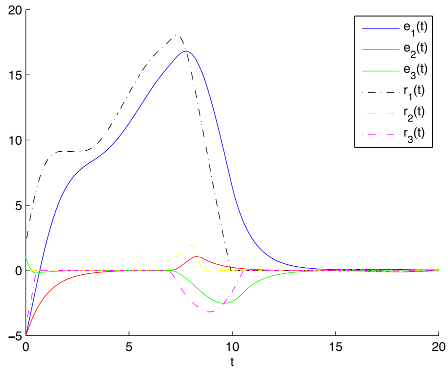

Figure 8. Curves of the error system after variable transform in Example 2.

Error curves in Fig. 8 converge to zero, confirming stable tracking. Thus, the drive and response systems achieve AS.

5 Conclusion

In this work, we discuss the AS for a class of MSINNs. Without using Lyapunov functional approach, MMMs, LMI, or integral inequality methods, two novel synchronization criteria of MSINNs are derived to guarantee the AS between the master and the slave systems by using the implicit zero point existence theorem and designing two new trigonometric controllers. In our study, the implicit zero point theorem of and the trigonometric controllers are jointly used to analyze AS. In future work, we will investigate GASN and finite-time synchronization for discrete-time NNs and fractional-order NNs.

Conflicts of interest. The authors declare no conflicts of interest.

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