Nonlinear Analysis: Modelling and Control, Vol. 31, No. 4, 938–957
https://doi.org/10.15388/namc.2026.31.47459

Approximate controllability of second-order impulsive integro-differential systems involving state-dependent delay and damping effects via resolvent operator

Raman Sasikumar, Velusamy Vijayakumar[1]

Department of Mathematics, School of Advanced Sciences,
Vellore Institute of Technology,
Vellore - 632 014, Tamil Nadu, India
sasikumardeivarama@gmail.com; vijaysarovel@gmail.com

Received: October 22, 2025 / **Revised:** April 16, 2026 / **Published online:** June 19, 2026

Abstract. This paper investigates the controllability of a second-order impulsive damped integro-differential nonautonomous system with state-dependent delay. The results are established using the properties of resolvent operators related to the second-order system behind them, and the analysis employs key mathematical tools such as Gronwall's lemma and fixed-point theorems. Assuming the approximate controllability of the corresponding linearized system, we obtain a new set of sufficient conditions developed for the approximate controllability of the nonlinear second-order system. These findings contribute to our understanding of the approximate controllability in complex dynamical systems affected by both damping and state-dependent delay. Finally, we present an application that demonstrates and validates the theoretical findings of this study.

Keywords: approximate controllability, integro-differential equations, resolvent operator, damping system, state-dependent delay.

1 Introduction

Controllability is a core concept in mathematical control theory, describing a qualitative characteristic of dynamic systems that determines whether their states can be guided to desired positions through suitable control inputs. It plays a pivotal role in both the theoretical analysis and practical design of control systems. Broadly speaking, controllability refers to the ability of a system to be steered from any initial state to any desired final state through a suitable choice of admissible controls. While much of the classical literature has focused on finite-dimensional systems, many challenging and unresolved questions remain for infinite-dimensional settings. Indeed, most criteria originally developed for finite-dimensional systems cannot be directly extended, which has motivated intensive research in this area over the past few decades. In infinite-dimensional spaces, exact controllability is often too restrictive and difficult to achieve, making approximate

[1]Corresponding author.

controllability a more practical and widely applicable notion. Under the assumption that the associated linear system is approximately controllable, a considerable body of recent work has been devoted to the study of such differential systems. For instance, the authors of [8] carried out a detailed investigation of second-order functional systems, focusing on their structural properties and dynamic behavior. Further, approximate controllability of integer- and fractional-order systems has been addressed in [25–27]. In various applications, it is often more beneficial to investigate second-order differential equations in their original form rather than transforming them into equivalent first-order systems. The foundational study of second-order evolution equations driven by the generator of a cosine family was initiated by [6], and later developments and extensions of this theory were carried out in [28].

Conversely, functional differential systems with state-dependent delay have gained popularity due to their utility in modeling various real-world phenomena. There are several important contributions in the literature. In [12, 19, 22, 23], solutions to first- and second-order partial functional differential equations with state-dependent delay were investigated. In [14], the controllability of second-order differential equations with state-dependent delay was investigated, while [24] analyzed the controllability of second-order systems with state-dependent delay and impulse. These studies emphasize both the theoretical and practical importance of incorporating state-dependent delay into functional differential equations. In recent years, authors have explored controllability results for neutral impulsive functional differential equations, as noted in [2]. Additionally, [24] investigated the approximate controllability of second-order impulsive systems with state-dependent delay. Such differential systems find extensive applications in various fields, including biological and mechanical models that are influenced by dynamic factors, electromagnetic disturbances, and impulsive shocks. For a thorough discussion on the effects of impulses, we refer to the monographs [4, 15] and the research studies [1, 18, 30].

In dynamical systems, damping is another important topic. It can be mathematically defined as a force synchronous with the object's velocity but directed in the opposite direction. Anomalous diffusion equations with damping are widely studied for modeling systems with memory and nonlocal effects, with practical applications such as mass dampers that stabilize bridges, buildings, and other structures. In this context, controllability and existence analysis for second-order damped systems have been extensively studied: [3] considered impulsive neutral systems with infinite delay; [10] examined abstract equations with impulses; and [2] addressed systems with state-dependent delay. In [9], the authors established the existence of solutions for second-order integro-differential equations and later extended it to neutral integro-differential equations with state-dependent delay in [17]. Further, [29] investigated controllability via resolvent operators. Motivated by these studies, this work derives suitable circumstances for the existence of a second-order impulsive damped integro-differential nonautonomous control system with state-dependent delay.

The major contributions of the present work may be highlighted as follows:

(i) Earlier studies have addressed related problems: Henríquez et al. [9] established the existence of solutions for nonautonomous second-order integro-differential equations, while Rezapour et al. [17] investigated the existence of second-order integro-differential nonautonomous systems with state-dependent delay. However, to the best of our knowledge, no work has investigated the approximate controllability of second-order damped impulsive integro-differential evolution systems with state-dependent delay via resolvent operators that also demonstrate damping behavior.

(ii) We establish the approximate controllability of second-order damped impulsive integro-differential evolution systems with state-dependent delay by employing resolvent operator techniques, thereby extending existing controllability results to a more general damped and delayed framework.

(iii) The analysis explicitly incorporates damping effects within the resolvent operator setting, providing new sufficient conditions that guarantee controllability for systems not covered by earlier nondamped or nonimpulsive models.

(iv) We employ the theory of strongly continuous cosine families of operators and resolvent operators to obtain the mild solution of system (1)–(3) and prove its existence using the fixed-point theorem.

(v) We establish a novel set of suitable circumstance that ensure the second-order impulsive damped abstract integro-differential evolution control system with state-dependent delay via resolvent operators under general assumptions.

(vi) Finally, the theoretical findings are substantiated and clarified through a comprehensive illustrative example.

In this work, we investigate the approximate controllability of a second-order impulsive damped integro-differential evolution system with state-dependent delay. To establish the framework of our analysis, we begin by formally defining the following system under consideration:

$$x''(t) = A(t)x(t) + \mathcal{D}x'(t) + \int_0^t \mathcal{E}(t,s)x(s)\,\mathrm{d}s + Bv(t) + F(t, x_{\rho(t,x_t)}), \quad t \in [0,b], \tag{1}$$

$$x_0 = \varphi \in \mathcal{B}, \qquad x'(0) = \eta \in \mathcal{Y}, \tag{2}$$

$$\Delta x(t_i) = \mathcal{J}_i^1(x_{t_i}), \quad \Delta x'(t_i) = \mathcal{J}_i^2(x_{t_i}), \quad i = 1, \dots, n. \tag{3}$$

Here $A(t) : D(A(t)) \subseteq \mathcal{Y} \to \mathcal{Y}$ and $\mathcal{E}(t,s) : D(\mathcal{E}) \subseteq \mathcal{Y} \to \mathcal{Y}$ are closed linear operators on a Banach space $\mathcal{Y}$, where the domain $D(\mathcal{E})$ is assumed to be independent of (t,s). The control function $v(\cdot)$ is taken from $L^2([0,b],U)$, where U is a Banach space. Moreover, $\mathcal{D}$ denotes a bounded linear operator on $\mathcal{Y}$. Further, we define the history segment $x_s : (-\infty, 0] \to \mathcal{Y}$ by $x_s(\theta) = x(s+\theta)$, which is an element of the phase space $\mathcal{B}$. In addition, we consider a nonlinear mapping $F : [0,b] \times \mathcal{B} \to \mathcal{Y}$, a delay functional $\rho : [0,b] \times \mathcal{B} \to (-\infty, b]$, and operators $\mathcal{J}_i^1, \mathcal{J}_i^2 : \mathcal{B} \to \mathcal{Y}$ for $i = 1, 2, \dots, n$. The notation $\Delta\xi(t)$ is used to denote the jump of ξ at t, which is assumed to be well defined for the system under consideration.

To cover the topic thoroughly, this document has several key sections.

(i) Section 2 introduces essential preliminary concepts, including fundamental definitions, useful lemmas, and a central theorem necessary for the later analysis.
(ii) In Section 3, we investigate the existence of a mild solution using fixed-point theory under appropriate assumptions, and a comprehensive investigation is carried out on approximate controllability of system (1)–(3), together with rigorous proofs supporting the obtained results.
(iii) Section 4 supports the theoretical findings with a detailed illustrative example. This example demonstrates the practical implementation and relevance of the derived results.

2 Preliminaries

This section outlines the essential definitions, notations, and preliminary lemmas required for the development of our main results. For convenience, we also introduce notational conventions that will be employed consistently throughout the paper. Let $(\mathcal{Y}, |\cdot|)$ be a Banach space, and consider the closed linear operators $A(t)$ and $\mathcal{E}(t,s)$ with domains $D(A)$ and $D(\mathcal{E})$, respectively, for $0 \leqslant s \leqslant t$. The domain $D(A(t))$, equipped with the graph norm induced by $A(t)$, is a Banach space. Furthermore, we suppose that all such norms under consideration are equivalent. The resolvent set of $A(t)$ is defined in the usual way, and the associated resolvent operator $(\lambda I - A(t))^{-1} : \mathcal{Y} \to D(A(t))$ is assumed to be well defined, bounded, and linear.

Nonautonomous second-order initial value problems have received a lot of attention due to their importance in modeling and analyzing complex dynamical systems. A typical example is given by the system

$$x''(t) = A(t)x(t) + f(t), \quad 0 \leqslant s, t \leqslant b, \tag{4}$$

$$x(s) = x^0, \qquad x'(s) = x^1. \tag{5}$$

Here $A(t) : D(A(t)) \subseteq \mathcal{Y} \to \mathcal{Y}$, $t \in [0,b]$, is a linear operator, and $f : t \in [0,b] \to \mathcal{Y}$ is a given function. In the existing literature, the analysis of solution existence for system (4)–(5) is often reduced to establishing the existence of an appropriate evolution operator $S(t,s)$ associated with the corresponding homogeneous system

$$x''(t) = A(t)x(t), \quad 0 \leqslant t \leqslant b.$$

Let $x \in D(A)$, and let the mapping $t \mapsto A(t)x$ be continuous. Furthermore, suppose that $A(\cdot)$ generates the two-parameter family $(S(t,s))_{0 \leqslant s \leqslant t \leqslant b}$ in the sense introduced by Henríquez [7, Def. 1.1]. For a detailed exposition of this concept and its analytical framework, we refer the reader to the aforementioned work. In the present context, it is sufficient to note that $S(\cdot)x$ is differentiable for every $x \in \mathcal{Y}$. Consequently, there exists a constant $M_1 > 0$ such that

$$\big\|S(t+h,\, s) - S(t,s)\big\| \leqslant M_1|h|$$

for all $s, t, t+h \in [0, b]$. We define the operator $C(t,s) = -\partial S(t,s)/\partial s$ and consider the function $f : t \in [0, b] \to \mathcal{Y}$. Next, we define the solution $x : [0, b] \to \mathcal{Y}$ of system (4)–(5) in the following form:

$$x(t) = C(t,s)x^0 + S(t,s)x^1 + \int_0^t S(t,\tau)f(\tau)\,\mathrm{d}\tau.$$

The following analysis focuses on second-order integro-differential system

$$x''(t) = A(t)x(t) + \int_0^t \mathcal{E}(t,\tau)x(\tau)\,\mathrm{d}\tau, \quad s \leqslant t \leqslant b, \tag{6}$$

$$x(s) = 0, \quad x'(s) = z \tag{7}$$

for $0 \leqslant s \leqslant b$. The issue being discussed has been explored in depth by [9], laying the groundwork for further exploration in this study. We denote $\Delta = \{(t,s)\colon 0 \leqslant s \leqslant t \leqslant b\}$.

We establish a set of conditions under which the operator $\mathcal{E}(\cdot)$ is well defined and possesses the desired analytical properties.

(Q_1) For every $0 \leqslant s \leqslant t \leqslant b$, the operator $\mathcal{E}(t,s) : [D(A)] \to \mathcal{Y}$ is bounded and linear. Moreover, for each $x \in D(A)$, the mapping $(t,s) \mapsto \mathcal{E}(t,s)x$ is continuous on $[0,b] \times [0,b]$, and

$$\big\|\mathcal{E}(t,s)x\big\| \leqslant c\|x\|_{[D(A)]}.$$

We assume that $c > 0$ is independent of $s, t \in \Delta$.

(Q_2) There exists a positive constant $L_{\mathcal{E}} > 0$ such that

$$\big\|\mathcal{E}(t_2,s)x - \mathcal{E}(t_1,s)x\big\| \leqslant L_{\mathcal{E}}\|t_2 - t_1\|_{[D(A)]}$$

for all $x \in D(A)$, $0 \leqslant s \leqslant t_1 \leqslant t_2 \leqslant b$.

(Q_3) There exists a constant $c_1 > 0$ such that

$$\left\| \int_\kappa^t S(t,s)\mathcal{E}(s,\kappa)x\,\mathrm{d}s \right\| \leqslant c_1\|x\|$$

for all $x \in D(A)$.

Under the specified conditions, there exists a resolvent operator $(\mathcal{V}(t,s))_{t\geqslant s}$ associated with system (6)–(7). In the subsequent analysis, we assume the existence of this operator and adopt its defining properties as the basis for our study.

Definition 1. (See [9].) A family of bounded linear operators $(\mathcal{V}(t,s))_{t\geqslant s}$ on the Banach space $\mathcal{Y}$ is called a resolvent operator associated with system (6)–(7) if it fulfills the following conditions:

(i) The mapping $\mathcal{V} : \Delta \to \mathcal{L}(\mathcal{Y})$ is strongly continuous. Moreover, for every $x \in \mathcal{Y}$, the mapping $\mathcal{V}(t,\cdot)x$ is continuously differentiable. In addition, $\mathcal{V}(s,s) = 0$, $\partial\mathcal{V}(t,s)/\partial t|_{t=s} = I$, and $\partial s\mathcal{V}(t,s)/\partial t|_{s=t} = -I$.

(ii) Assume that $z \in D(A)$. Then the function $\mathcal{V}(\cdot, s)z$ is a solution of system (6)–(7). In other words,

$$\frac{\partial^2}{\partial t^2}\mathcal{V}(t,\nu)z = A(t)\mathcal{V}(t,s)z + \int_s^t \mathcal{E}(t,\tau)\mathcal{V}(\tau,s)z\,\mathrm{d}\tau, \quad 0 \leqslant s \leqslant t \leqslant b.$$

From condition (i) it can be deduced that there exist $P, \tilde{P} > 0$ such that

$$\|\mathcal{V}(t,s)\| \leqslant P, \quad \left\|\frac{\partial}{\partial s}\mathcal{V}(t,s)\right\| \leqslant \tilde{P}, \quad (t,s) \in \Delta.$$

Furthermore, we consider the associated linear operator

$$G(t,\kappa)z = \int_\kappa^t \mathcal{E}(t,s)\mathcal{V}(s,\kappa)z\,\mathrm{d}s, \quad z \in D(A),\ 0 \leqslant \kappa \leqslant t \leqslant b.$$

The construction can be naturally extended to the space $\mathcal{Y}$. We denote this extension by the same symbol $G(t,\kappa)$, where $G : \Delta \to \mathcal{L}(H)$ is a strongly continuous operator family. Moreover, it is rigorously established that

$$\mathcal{V}(t,\kappa)z = S(t,\kappa) + \int_\kappa^t S(t,s)G(t,\kappa)z\,\mathrm{d}s, \quad z \in \mathcal{Y}. \tag{8}$$

From this property it follows that $\mathcal{V}(\cdot)$ satisfies a uniform Lipschitz condition. In other words, we assume that there exists a constant $L_{\mathcal{V}} > 0$ such that

$$\|\mathcal{V}(t+h,\,\kappa) - \mathcal{V}(t,\kappa)\| = L_{\mathcal{V}}|h|, \quad t,\, t+h,\, \kappa \in [0,b].$$

Let $g : \mathcal{V} \to \mathcal{Y}$ be an integrable mapping. We now consider the corresponding non-homogeneous system

$$x''(t) = A(t)x(t) + \int_0^t \mathcal{E}(t,s)x(s)\,\mathrm{d}s + g(t), \quad t \in [0,b], \tag{9}$$

$$x(0) = z^0, \qquad x'(0) = z^1. \tag{10}$$

Then mild solution corresponding to system (9)–(10), previously discussed in [9], is now introduced for further analysis.

Definition 2. (See [9].) Let $z^0, z^1 \in \mathcal{Y}$. The function $x : [0,b] \to \mathcal{Y}$ defined by

$$x(t) = -\frac{\partial\mathcal{V}(t,0)z^0}{\partial t} + \mathcal{V}(t,0)z^1 + \int_0^t \mathcal{V}(t,s)g(s)\,\mathrm{d}s$$

is called the mild solution corresponding to system (9)–(10).

If the function ϑ is piecewise continuous on the interval $(\sigma, a]$, then we say that $\vartheta : [\sigma, a] \to \mathcal{Y}$ is a normalized piecewise continuous function on $[\sigma, a]$. In particular, we consider the space $\mathbb{PC}$ consisting of all piecewise continuous functions $\vartheta : [\sigma, a] \to \mathcal{Y}$ that are normalized in the sense that ϑ is defined for all $t \neq t_i$, $i = 1, \dots, n$. Equipped with the norm of uniform convergence, this space is a Banach space.

Consider the partition defined by $t_0 = 0$ and $t_{n+1} = b$. For $\vartheta \in \mathbb{PC}$, we define $\tilde{\vartheta}_i$ as follows. For every $i = 0, 1, \dots, n$, the function $\tilde{\vartheta}_i$ belongs to the space $\mathbb{C}([t_i, t_{i+1}]; \mathcal{Y})$ and is given by

$$\tilde{\vartheta}_i(t) = \begin{cases} \vartheta(t) & \text{for } t \in (t_i, t_{i+1}], \\ \vartheta(t_i^+) & \text{for } t = t_i. \end{cases}$$

Additionally, we introduce the set $\tilde{\mathscr{B}} = \{\tilde{\vartheta}_i,\ \vartheta \in \mathscr{B}\}$.

Lemma 1. (*See* [11].) *If, for each $i = 0, 1, \dots, n$, the set $\tilde{\mathscr{B}}_i$ is relatively compact in $\mathbb{C}([t_i, t_{i+1}], \mathcal{Y})$, then the collection $\mathscr{B} \subseteq \mathbb{PC}$ is also relatively compact in the space $\mathbb{PC}$.*

In this study, we adopt an axiomatic formulation of the phase space $\mathcal{B}$, following the approach introduced in [13]. Concretely, $\mathcal{B}$ is considered as a linear space of functions defined on $(-\infty, 0]$ with values in $\mathcal{Y}$, equipped with a seminorm $|\cdot|_{\mathcal{B}}$ that satisfies the following axioms:

(A) Let $x : (-\infty, \sigma + b] \to \mathcal{Y}$ with $b > 0$. Then the restriction of x to the interval $[\sigma, \sigma + b]$ belongs to the space of piecewise continuous functions, that is, $x|_{[\sigma, \sigma+b]} \in \mathbb{PC}([\sigma, \sigma + b]; \mathcal{Y})$, and its initial segment x_σ lies in $\mathcal{B}$. Moreover, for each $t \in [\sigma, \sigma + b]$, the following properties hold under the specified conditions:
  - (i) $x_t \in \mathcal{B}$.
  - (ii) $\|x_t\| \leqslant \mathcal{H}\|x_t\|_{\mathcal{B}}$.
  - (iii) $\|x_t\|_{\mathcal{B}} \leqslant \mathcal{K}(t - \sigma) \sup\{\|x(\mathfrak{d})\|,\ \sigma \leqslant \mathfrak{d} \leqslant t\} + \mathscr{M}(t - \sigma)\|x_\sigma\|_{\mathcal{B}}$, where $\mathcal{H} > 0$ is a fixed constant, and $\mathcal{K}, \mathscr{M} : [0, \infty) \to [1, \infty)$ are functions independent of $x(\cdot)$. Under these conditions, the function $\mathscr{M}$ is locally bounded.

(B) The space $\mathcal{B}$ is complete.

By Definition 2, the mapping $x(\cdot)$ is well defined and continuous. It has been established in [20] that the stated assumption holds under the given conditions.

(H_0) As $\alpha \to 0^+$, the operator sequence $\alpha R(\alpha, \Gamma_0^b)$ converges strongly to the zero operator.

Building on the results of [16], if condition (H_0) is satisfied, then the corresponding linear system

$$x''(t) = Ax(t) + Bv(t), \quad t \in [0, b],$$
$$x(0) = \varphi, \qquad x'(0) = \eta$$

is approximately controllable on $[0, b]$.

Definition 3. A continuous mapping $x:(-\infty,b]\to\mathcal{Y}$ is said to be a solution of system (1)–(3) if it satisfies the initial condition $x_0=\varphi$ and the following integral equation:

$$\begin{aligned} x(t) = {}& -\frac{\partial\mathcal{V}(t,s)\varphi(0)}{\partial s}\bigg|_{s=0} + \mathcal{V}(t,0)\eta - \int_0^t \frac{\partial\mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s \\ & + \int_0^t \mathcal{V}(t,s)Bv(s)\,\mathrm{d}s + \sum_{i=0}^{j-1}\big[\mathcal{V}(t,t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t,t_i)\mathcal{D}x(t_i^+)\big] \\ & - \mathcal{V}(t,t_j)\mathcal{D}x(t_j^+) + \int_0^t \mathcal{V}(t,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s \\ & - \sum_{t_i<t}\frac{\partial\mathcal{V}(t,s)}{\partial s}\mathcal{J}_i^1(x_{t_i}) + \sum_{t_i<t}\mathcal{V}(t,s)\mathcal{J}_i^2(x_{t_i}), \quad t\in[0,b]. \end{aligned}$$

3 Existence of mild solutions

This section is devoted to studying the conditions that guarantee the existence of mild solutions via Schauder's fixed-point theorem for the abstract system given by (1)–(3). To this end, we assume that $\rho:[0,b]\times\mathcal{B}\to(-\infty,b]$ is a continuous function. Furthermore, we impose the following assumptions, which will play a crucial role in establishing our main results.

(H$_1$) The mapping $t\to\phi_t$ is well defined and continuous on the set $\mathscr{Z}(\rho^-)=\{\rho(s,\xi)\colon(s,\xi)\in[0,b]\times\mathcal{B},\ \rho(s,\xi)\leqslant 0\}$. Furthermore, there exists a continuous and bounded function $J^\phi:\mathscr{Z}(\rho^-)\to(0,\infty)$ such that, for each $t\in\mathscr{Z}(\rho^-)$, the following inequality holds:

$$\|\phi_t\|_{\mathcal{B}}\leqslant J^\phi(t)\|\phi\|_{\mathcal{B}}.$$

(H$_2$) The mapping $F:[0,b]\times\mathcal{B}\to\mathcal{Y}$ satisfies the conditions outlined below:

- (i) For each $\xi\in\mathcal{B}$, $F(\cdot,\xi):[0,b]\to\mathcal{Y}$ is strongly measurable.
- (ii) For every $t\in[0,b]$, $F(t,\cdot):\mathcal{B}\to\mathcal{Y}$ is continuous.
- (iii) For each $r>0$, there exists a function $\lambda_r\in L^1([0,b],\mathbb{R}^+)$ such that

$$\sup_{\|\xi\|\leqslant r}\big\|F(t,\xi)\big\|\leqslant\lambda_r(t),\quad\text{a.e. } t\in[0,b],$$

and

$$\lim_{r\to\infty}\inf\int_0^b\frac{\lambda_r(t)}{r}\,\mathrm{d}t=\gamma<\infty.$$

(H$_3$) The operators $\mathcal{J}_i^1,\mathcal{J}_i^2:\mathcal{B}\to\mathcal{Y}$ are continuous, and there exist positive constants c_i^1 and c_i^2, $i=1,2,\dots,n$, such that

$$\big\|\mathcal{J}_i^1(\xi)\big\|\leqslant c_i^1\quad\text{and}\quad\big\|\mathcal{J}_i^2(\xi)\big\|\leqslant c_i^2,\quad i=1,2,\dots,n,\ \xi\in\mathcal{B}.$$

Lemma 2. *Let $x : (-\infty, b] \to \mathcal{Y}$ be a function such that $x_0 = \varphi$. Then*

$$\|x\|_{\mathcal{B}} \leqslant \big(\mathscr{M}_b + J_0^{\phi}\big)\|\phi\|_{\mathcal{B}} + \mathcal{K}_b \sup\big\{\big\|x(\theta)\big\|, \theta \in \big[0, \max\{0, s\}\big]\big\}$$

for $s \in \mathscr{Z}(\rho^-) \cup [0, b]$. Here $J_0^{\phi} = \sup_{t \in \mathscr{Z}(\rho^-)} J^{\phi}(t)$, and $\mathscr{Z}(\rho^-)$ denotes the set of discontinuity points prior to ρ, while $[0, b]$ represents the corresponding interval of continuity.

Consider the function space $\mathcal{P} = \{x \in \mathbb{PC}([0, b], \mathcal{Y})\colon x(0) = \varphi(0)\}$. On this space, we define the bounded subset $\mathscr{B}_r = \{x \in \mathcal{P}\colon \|x\| \leqslant r\}$, where $r > 0$. Within this framework, we show that the dynamical system given by (1)–(3) is controllable, provided that for every $\alpha > 0$, one can find a continuous trajectory $x(\cdot) \in \mathcal{P}$ satisfying the required conditions. The function is given by

$$\begin{aligned} x(t) = {}& -\frac{\partial \mathcal{V}(t, s)\varphi(0)}{\partial s}\bigg|_{s=0} + \mathcal{V}(t, 0)\eta - \int_0^t \frac{\partial \mathcal{V}(t, s)}{\partial s} \mathcal{D}x(s)\,\mathrm{d}s \\ & + \int_0^t \mathcal{V}(t, s)Bv(s)\,\mathrm{d}s + \sum_{i=0}^{j-1} \big[\mathcal{V}(t, t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t, t_i)\mathcal{D}x(t_i^+)\big] \\ & - \mathcal{V}(t, t_j)\mathcal{D}x(t_j^+) + \int_0^t \mathcal{V}(t, s)F\big(s, x_{\rho(s, x_s)}\big)\,\mathrm{d}s \\ & - \sum_{t_i < t} \frac{\partial \mathcal{V}(t, s)}{\partial s} \mathcal{J}_i^1(x_{t_i}) + \sum_{t_i < t} \mathcal{V}(t, s)\mathcal{J}_i^2(x_{t_i}), \quad t \in [0, b], \\ v(s, x) = {}& B^* \mathcal{V}^*(b, t) R\big(\alpha, \Gamma_0^b\big) q\big(x(\cdot)\big), \end{aligned}$$

where

$$\begin{aligned} q(x(\cdot)) = {}& x_b + \frac{\partial \mathcal{V}(t, s)\varphi(0)}{\partial s}\bigg|_{s=0} - \mathcal{V}(t, 0)\eta + \int_0^t \frac{\partial \mathcal{V}(t, s)}{\partial s} \mathcal{D}x(s)\,\mathrm{d}s \\ & - \sum_{i=0}^{j-1} \big[\mathcal{V}(t, t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t, t_i)\mathcal{D}x(t_i^+)\big] + \mathcal{V}(t, t_j)\mathcal{D}x(t_j^+) \\ & - \int_0^t \mathcal{V}(t, s)F(s, x_{\rho(s, x_s)})\,\mathrm{d}s + \sum_{t_i < t} \frac{\partial \mathcal{V}(t, s)}{\partial s} \mathcal{J}_i^1(x_{t_i}) - \sum_{t_i < t} \mathcal{V}(t, s)\mathcal{J}_i^2(x_{t_i}). \end{aligned}$$

We are now in a position to establish the existence of the desired result.

Theorem 1. *Assuming that conditions* (H_1)–(H_3) *are fulfilled. Then system* (1)–(3) *admits a solution on the interval* $[0, b]$*, provided that the following additional requirements are satisfied*:

$$\left(1 + \frac{1}{\alpha} b P^2 \mathcal{M}_B^2\right)\big(3P\|\mathcal{D}\| + b\tilde{P}\|\mathcal{D}\| + P\gamma\mathcal{K}_b\big) < 1, \qquad \|B\| \leqslant \mathcal{M}_B.$$

Proof. Let $\mathcal{P}$ be a given space. Then

$$\mathscr{B}_r = \{x \in \mathcal{P}\colon \|x\| \leqslant r\}, \quad r > 0.$$

We define the mapping $\Phi : \mathcal{P} \to \mathcal{P}$ as follows:

$$\begin{aligned}
(\Phi x)(t) = {}& -\frac{\partial \mathcal{V}(t,s)\varphi(0)}{\partial s}\bigg|_{s=0} + \mathcal{V}(t,0)\eta - \int_0^t \frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s \\
& + \int_0^t \mathcal{V}(t,s)Bv(s)\,\mathrm{d}s + \sum_{i=0}^{j-1}\big[\mathcal{V}(t,t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t,t_i)\mathcal{D}x(t_i^+)\big] \\
& - \mathcal{V}(t,t_j)\mathcal{D}x(t_j^+) + \int_0^t \mathcal{V}(t,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s \\
& - \sum_{t_i<t}\frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{J}_i^1(x_{t_i}) + \sum_{t_i<t}\mathcal{V}(t,s)\mathcal{J}_i^2(x_{t_i}), \quad t \in [0,b].
\end{aligned}$$

It can be established that, for every $\alpha > 0$, the mapping $\Phi : \mathcal{P} \to \mathcal{P}$ admits at least one fixed point.

Step 1. For any fixed $\alpha > 0$, one can assert the existence of some $r > 0$ such that $\Phi(\mathscr{B}_r) \subset \mathscr{B}_r$. To prove this, assume by contradiction that no such r exists. Then, for this choice of $\alpha > 0$, it follows that for every $r > 0$, there exist $x \in \mathscr{B}_r$ and $t \in [0,b]$ such that $r < \|\Phi x(t)\|$.

For any positive value of $\alpha > 0$, we observe that

$$\begin{aligned}
r &< \big\|(\Phi x)(t)\big\| \\
&\leqslant \left\|\frac{\partial \mathcal{V}(t,s)\varphi(0)}{\partial s}\bigg|_{s=0}\right\| + \big\|\mathcal{V}(t,0)\eta\big\| + \left\|\int_0^t \frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s\right\| \\
&\quad + \left\|\int_0^t \mathcal{V}(t,s)Bv(s)\,\mathrm{d}s\right\| \\
&\quad + \left\|\sum_{i=0}^{j-1}\big[\mathcal{V}(t,t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t,t_i)\mathcal{D}x(t_i^+)\big] - \mathcal{V}(t,t_j)\mathcal{D}x(t_j^+)\right\| \\
&\quad + \left\|\int_0^t \mathcal{V}(t,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s\right\| + \left\|\sum_{t_i<t}\frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{J}_i^1(x_{t_i})\right\| \\
&\quad + \left\|\sum_{t_i<t}\mathcal{V}(t,s)\mathcal{J}_i^2(x_{t_i})\right\|
\end{aligned}$$

$$\leqslant \tilde{P}\mathcal{H}\|\varphi\|_{\mathcal{B}} + P\|\eta\| + 3P\|\mathcal{D}\|r + \tilde{P}\int_0^t \|\mathcal{D}\|\|x(s)\|\,\mathrm{d}s + P\mathcal{M}_B\int_0^t \|v(s)\|\,\mathrm{d}s$$
$$+ P\int_0^t \|F(s, x_{\rho(s,x_s)})\|\,\mathrm{d}s + \tilde{P}\sum_{i=1}^n \|\mathcal{J}_i^1(x_{t_i})\| + P\sum_{i=1}^n \|\mathcal{J}_i^2(x_{t_i})\|.$$

For every $x \in \mathscr{B}_r$, one can deduce from Lemma 2 that

$$\|x_{\rho(s,x_s)}\| \leqslant (\mathscr{M}_b + J_0^\phi)\|\phi\|_{\mathcal{B}} + \mathcal{K}_b r = r^*,$$

where r^* denotes a positive constant. Therefore, we arrive at the following result:

$$r \leqslant \tilde{P}\mathcal{H}\|\varphi\|_{\mathcal{B}} + P\|\eta\| + 3P\|\mathcal{D}\|r + b\tilde{P}\|\mathcal{D}\|r$$
$$+ \frac{1}{\alpha}bP^2\mathcal{M}_B^2\Biggl(\|x_b\| + \tilde{P}\mathcal{H}\|\varphi\|_{\mathcal{B}} + P\|\eta\| + 3P\|\mathcal{D}\|r + b\tilde{P}\|\mathcal{D}\|r$$
$$+ P\int_0^b \lambda_{r^*}(s)\,\mathrm{d}s + \tilde{P}\sum_{i=1}^n c_i^1 + P\sum_{i=1}^n c_i^2\Biggr) + P\int_0^b \lambda_{r^*}(s)\,\mathrm{d}s + \tilde{P}\sum_{i=1}^n c_i^1 + P\sum_{i=1}^n c_i^2.$$

Dividing the expression by r and letting $r^* \to \infty$ as $r \to \infty$, we arrive at the following limiting relation:

$$\lim_{r\to\infty}\inf\int_0^b \frac{\lambda_{r^*}(s)}{r}\,\mathrm{d}s = \lim_{r\to\infty}\inf\int_0^b \left(\frac{\lambda_{r^*}(s)}{r^*}\cdot\frac{r^*}{r}\right)\mathrm{d}s = \gamma\mathcal{K}_b.$$

Hence, for $\alpha > 0$,

$$\left(1 + \frac{1}{\alpha}bP^2\mathcal{M}_B^2\right)\left(3P\|\mathcal{D}\| + b\tilde{P}\|\mathcal{D}\| + P\gamma\mathcal{K}_b\right) \geqslant 1,$$

which is a contradiction to our assumption. Consequently, for any $\alpha > 0$, there exists $r > 0$ such that the operator Φ maps $\mathscr{B}_r$ into itself.

Step 2. For each $\alpha > 0$, the operator Φ maps $\mathscr{B}_r$ into a relatively compact subset of $\mathscr{B}_r$. To begin, we show that the set $\Upsilon(t) = \{\Phi x(t),\ x \in \mathscr{B}_r\}$ is relatively compact in $\mathcal{Y}$. The case $t = 0$ is obvious.

For $0 < \epsilon < t \leqslant b$, define

$$(\Phi^\epsilon x)(t) = -\frac{\partial\mathcal{V}(t,s)\varphi(0)}{\partial s}\bigg|_{s=0} + \mathcal{V}(t,0)\eta - \int_0^{t-\epsilon}\frac{\partial\mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s$$
$$+ \int_0^{t-\epsilon}\mathcal{V}(t,s)Bv(s)\,\mathrm{d}s + \sum_{i=0}^{j-1}\left[\mathcal{V}(t,t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t,t_i)\mathcal{D}x(t_i^+)\right]$$

$$- \mathcal{V}(t,t_j)\mathcal{D}x(t_j^+) + \int_0^{t-\epsilon} \mathcal{V}(t,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s$$
$$- \sum_{t_i<t} \frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{J}_i^1(x_{t_i}) + \sum_{t_i<t} \mathcal{V}(t,s)\mathcal{J}_i^2(x_{t_i}).$$

Because $\mathcal{V}(t)$ is compact, the set $\Upsilon_\epsilon(t) = \{(\Phi^\epsilon x)(t),\, x(\cdot) \in \mathscr{B}_r\}$ is relatively compact in $\mathcal{Y}$.

On the other hand,

$$\begin{aligned}
&\big\|(\Phi x)(t) - \big(\Phi^\epsilon x\big)(t)\big\| \\
&\quad = \left\| -\int_{t-\epsilon}^{t} \frac{\partial \mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s + \int_{t-\epsilon}^{t} \mathcal{V}(t,s)Bv(s)\,\mathrm{d}s + \int_{t-\epsilon}^{t} \mathcal{V}(t,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s \right\| \\
&\quad \leqslant \epsilon\tilde{P}\|\mathcal{D}\|r + \frac{1}{\alpha}\epsilon P^2\mathcal{M}_B^2\Bigg(\|x_b\| + \tilde{P}\mathcal{H}\|\varphi\|_{\mathcal{B}} + P\|\eta\| + 3P\|\mathcal{D}\|r + b\tilde{P}\|\mathcal{D}\|r \\
&\qquad + P\int_0^b \lambda_{r^*}(s)\,\mathrm{d}s + \tilde{P}\sum_{i=1}^n c_i^1 + P\sum_{i=1}^n c_i^2 \Bigg) + P\int_{t-\epsilon}^{t} \lambda_{r^*}(s)\,\mathrm{d}s.
\end{aligned}$$

Hence for each $t \in [0,b]$, $\Upsilon(t)$ is relatively compact in $\mathcal{Y}$.

Next, we show that the set $\Upsilon = \{(\Phi x)(\cdot),\, x(\cdot) \in \mathscr{B}_r\}$ is an equicontinuous family of functions on the interval $[0,b]$. For $0 < t_1 < t_2 \leqslant b$,

$$\begin{aligned}
&\big\|(\Phi x)(t_2) - (\Phi x)(t_1)\big\| \\
&\quad \leqslant \left\| \frac{\partial \mathcal{V}(t_2,s)\varphi(0)}{\partial s}\bigg|_{s=0} - \frac{\partial \mathcal{V}(t_1,s)\varphi(0)}{\partial s}\bigg|_{s=0} \right\| + \big\|\mathcal{V}(t_2,0)\eta - \mathcal{V}(t_1,0)\eta\big\| \\
&\qquad + \left\| \int_0^{t_1} \left[\frac{\partial \mathcal{V}(t_2,s)}{\partial s} - \frac{\partial \mathcal{V}(t_1,s)}{\partial s} \right]\mathcal{D}x(s)\,\mathrm{d}s \right\| + \left\| \int_{t_1}^{t_2} \frac{\partial \mathcal{V}(t_2,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s \right\| \\
&\qquad + \left\| \int_0^{t_1} \big[\mathcal{V}(t_2,s) - \mathcal{V}(t_1,s)\big]Bv(s)\,\mathrm{d}s \right\| + \left\| \int_{t_1}^{t_2} \mathcal{V}(t_2,s)Bv(s)\,\mathrm{d}s \right\| \\
&\qquad + \left\| \sum_{i=0}^{j-1} \big[\big(\mathcal{V}(t_2,t_{i+1}) - \mathcal{V}(t_1,t_{i+1})\big)\mathcal{D}x(t_{i+1}^-) - \big(\mathcal{V}(t_2,t_i) - \mathcal{V}(t_1,t_i)\big)\mathcal{D}x(t_i^+) \big] \right\| \\
&\qquad + \big\| \big[\mathcal{V}(t_2,t_j) - \mathcal{V}(t_1,t_j)\big]\mathcal{D}x(t_j^+) \big\| + \left\| \int_0^{t_1} \big[\mathcal{V}(t_2,s) - \mathcal{V}(t_1,s)\big]F(s,x_{\rho(s,x_s)})\,\mathrm{d}s \right\|
\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_{t_1}^{t_2}\mathcal{V}(t_2,s)F(s,x_{\rho(s,x_s)})\,\mathrm{d}s\right\|+\left\|\sum_{t_i<t}\left[\frac{\partial\mathcal{V}(t_2,s)}{\partial s}-\frac{\partial\mathcal{V}(t_1,s)}{\partial s}\right]\mathcal{J}_i^1(x_{t_i})\right\|\\
&+\left\|\sum_{t_i<t}\left[\mathcal{V}(t_2,s)-\mathcal{V}(t_1,s)\right]\mathcal{J}_i^2(x_{t_i})\right\|\\
\leqslant{}&\left\|\frac{\partial\mathcal{V}(t_2,s)}{\partial s}\bigg|_{s=0}-\frac{\partial\mathcal{V}(t_1,s)}{\partial s}\bigg|_{s=0}\right\|\mathcal{H}\|\varphi\|_{\mathcal{B}}+\left\|\mathcal{V}(t_2,0)-\mathcal{V}(t_1,0)\right\|\|\eta\|\\
&+\int_0^{t_1}\left\|\frac{\partial\mathcal{V}(t_2,s)}{\partial s}-\frac{\partial\mathcal{V}(t_1,s)}{\partial s}\right\|\|\mathcal{D}\|r\,\mathrm{d}s+\tilde{P}\|\mathcal{D}\|r(t_2-t_1)\\
&+\mathcal{M}_B\int_0^{t_1}\left\|\mathcal{V}(t_2,s)-\mathcal{V}(t_1,s)\right\|\left\|v(s)\right\|\mathrm{d}s+P\mathcal{M}_B\int_{t_1}^{t_2}\left\|v(s)\right\|\mathrm{d}s\\
&+\sum_{i=0}^{j-1}\left[\left\|\mathcal{V}(t_2,t_{i+1})-\mathcal{V}(t_1,t_{i+1})\right\|\|\mathcal{D}\|r+\left\|\mathcal{V}(t_2,t_i)-\mathcal{V}(t_1,t_i)\right\|\|\mathcal{D}\|r\right]\\
&+\left\|\mathcal{V}(t_2,t_j)-\mathcal{V}(t_1,t_j)\right\|\|\mathcal{D}\|r+P\int_0^{t_1}\left\|\mathcal{V}(t_2,s)-\mathcal{V}(t_1,s)\right\|\lambda_{r^*}(s)\,\mathrm{d}s\\
&+P\int_{t_1}^{t_2}\lambda_{r^*}(s)\,\mathrm{d}s+\sum_{i=1}^{n}\left\|\frac{\partial\mathcal{V}(t_2,s)}{\partial s}-\frac{\partial\mathcal{V}(t_1,s)}{\partial s}\right\|c_i^1\\
&+\sum_{i=1}^{n}\left\|\mathcal{V}(t_2,s)-\mathcal{V}(t_1,s)\right\|c_i^2.
\end{aligned}
$$

The expression on the right-hand side is independent of the specific choice of $x(\cdot)\in\mathscr{B}_r$ and converges to zero as $t_2-t_1\to 0$. This follows from the fact that the compactness of $\mathcal{V}(t)$ for $t>0$ ensures continuity with respect to the uniform topology of operators. This establishes that $\varUpsilon$ is right-equicontinuous at each $t\in(0,b)$. The arguments for right-equicontinuity at $t=0$ and for left-equicontinuity at points $t\in(0,b]$ proceed in a similar manner. Consequently, the set $\varPhi(\mathscr{B}_r)$ is equicontinuous on $[0,b]$.

Step 3. The mapping $\varPhi(\cdot)$ exhibits continuity on $\mathscr{B}_r$.

Let x^n, $n\in\mathbb{N}$, be a sequence in $\mathscr{B}r$, and let $x\in\mathscr{B}_r$ be such that $x^n\to x$ with respect to the topology of $\mathcal{P}$. By invoking axiom (A), the following result emerges: $x^n_{\rho(s,x^n_s)}\to x_{\rho(s,x_s)}$ as $n\to\infty$ for every $s\in[0,b]$.

Now, from the inequality

$$
\begin{aligned}
&\left\|F\left(s,x^n_{\rho(s,x^n_s)}\right)-F(s,x_{\rho(s,x_s)})\right\|\\
&\quad\leqslant\left\|F(s,x^n_{\rho(s,x^n_s)})-F(s,x_{\rho(s,x^n_s)})\right\|+\left\|F(s,x_{\rho(s,x^n_s)})-F(s,x_{\rho(s,x_s)})\right\|
\end{aligned}
$$

we infer that $F(s, x^n_{\rho(s,x^n_s)}) \to F(s, x_{\rho(s,x_s)})$ as $n \to \infty$ for all $s \in [0, b]$. By invoking the Lebesgue dominated convergence theorem, it follows that $\varPhi x^n \to \varPhi x$ in $\mathcal{P}$. Consequently, the operator $\varPhi(\cdot)$ is continuous on $\mathscr{B}_r$. Therefore, applying Schauder's fixed-point theorem, we deduce that $\varPhi$ admits at least one fixed point, which ensures the existence of a solution to problem (1) on $[0, b]$. □

Definition 4. The second-order control system (1)–(3) is said to be approximately controllable on the interval $[0, b]$ if the closure of its reachable set coincides with the entire state space, that is, $\overline{R(b, x_0)} = \mathcal{Y}$. Here the reachable set is given by $R(b, x_0) = \{x_b(x_0; v),\ v(\cdot) \in L^2([0, b], U)\}$, where $x_b(x_0; v)$ denotes the mild solution of system (1)–(3) corresponding to the initial state x_0 and control input $v(\cdot)$. In other words, for approximate controllability, it must be possible to steer the system arbitrarily close to any target state in $\mathcal{Y}$ using admissible control functions.

Theorem 2. *Assuming that conditions* (H$_0$)–(H$_3$) *are satisfied and the family $\mathcal{V}(t)$, $t > 0$, is compact, it follows that system* (1)–(3) *is approximately controllable over the interval* $[0, b]$.

Proof. Let $x^\alpha(\cdot) = \varPhi(x)(\cdot)$ for $x \in \mathscr{B}_r$. According to Theorem 1, every fixed point of $\varPhi$ corresponds to a mild solution of system (1)–(3) under the prescribed control

$$v^\alpha(t) = B^*\mathcal{V}^*(b, t)R\big(\alpha, \varGamma_0^b\big)q\big(x^\alpha\big),$$

while fulfilling the inequality

$$x^\alpha(b) = x_b + \alpha R\big(\alpha, \varGamma_0^b\big)q\big(x^\alpha\big). \tag{11}$$

Since the operator F is bounded, one can assert the existence of $\mathfrak{L} > 0$ such that

$$\int_0^b \big\|F\big(s, x^\alpha_{\rho(s,x_s)}\big)\big\| \leqslant \mathfrak{L}b,$$

and consequently, the sequence $\{F(s, x^\alpha_{\rho(s,x^\alpha_s)})\}$ is bounded in $L^2([0, b], \mathcal{Y})$. Hence, there is a subsequence, still denoted by $\{F(s, x^\alpha_{\rho(s,x^\alpha_s)})\}$, which converges weakly to some function $f(s)$ in $L^2([0, b], \mathcal{Y})$. Now, we define

$$\begin{aligned}\omega = {}& -\frac{\partial\mathcal{V}(t,s)\varphi(0)}{\partial s}\bigg|_{s=0} + \mathcal{V}(t,0)\eta - \int_0^t \frac{\partial\mathcal{V}(t,s)}{\partial s}\mathcal{D}x(s)\,\mathrm{d}s + \int_0^t \mathcal{V}(t,s)Bv(s)\,\mathrm{d}s \\ & + \sum_{i=0}^{j-1}\big[\mathcal{V}(t,t_{i+1})\mathcal{D}x(t_{i+1}^-) - \mathcal{V}(t,t_i)\mathcal{D}x(t_i^+)\big] - \mathcal{V}(t,t_j)\mathcal{D}x(t_j^+) \\ & + \int_0^t \mathcal{V}(t,s)f(s)\,\mathrm{d}s - \sum_{t_i<t}\frac{\partial\mathcal{V}(t,s)}{\partial s}\mathcal{J}_i^1(x_{t_i}) + \sum_{t_i<t}\mathcal{V}(t,s)\mathcal{J}_i^2(x_{t_i}) - x_b.\end{aligned}$$

Then, we have

$$\begin{aligned}\|q(x^\alpha)-\omega\| &= \left\|\int_0^b \mathcal{V}(b,s)\big[F(s,x^\alpha_{\rho(s,x^\alpha_s)})-f(s)\big]\,\mathrm{d}s\right\| \\ &\leqslant \sup_{t\in[0,b]}\left\|\int_0^t \mathcal{V}(t-s)\big[F(s,x^\alpha_{\rho(s,x^\alpha_s)})-f(s)\big]\,\mathrm{d}s\right\| \to 0 \qquad (12)\end{aligned}$$

as $\alpha \to 0^+$. By the infinite-dimensional Arzelà–Ascoli theorem, the operator $\ell(\cdot) \to \int_0^\cdot \mathcal{V}(t,s)\ell(s)\,\mathrm{d}s : L^2([0,b],\mathcal{Y}) \to \mathcal{C}([0,b],\mathcal{Y})$ is compact. Hence, the right-hand side of (12) tends to 0 as $\alpha \to 0^+$. Furthermore, from (11) we obtain

$$\begin{aligned}\|x^\alpha(b)-x_b\| &\leqslant \|\alpha R(\alpha,\Gamma_0^b)(\omega)\| + \|\alpha R(\alpha,\Gamma_0^b)\|\|q(x^\alpha)-\omega\| \\ &\leqslant \|\alpha R(\alpha,\Gamma_0^b)(\omega)\| + \|q(x^\alpha)-\omega\| \to 0\end{aligned}$$

as $\alpha \to 0^+$. This show the approximate controllability of the given system (1)–(3), thereby completing the proof. □

Remark 1. Differential inclusions and second-order neutral differential inclusions arise in many areas of applied mathematics. The analytical technique developed in this work can be naturally extended to investigate the approximate controllability of nonlinear second-order neutral nonautonomous evolution inclusions with damping effects via the resolvent operator framework. In particular, by suitably introducing and adapting the multivalued maps proposed in [5, 21], one can derive approximate controllability results for a broad class of second-order differential inclusion systems with damping.

4 An example

This part is devoted to demonstrating the effective use of the theoretical framework established in the preceding sections, focusing on the wave propagation in a bar defined over the interval $[0,\pi]$ with fixed boundary conditions. In particular, we consider a damped impulsive second-order partial integro-differential equation incorporating a state-dependent delay and a control function $\hat{\mu}(t,\cdot) \in L^2[0,\pi]$, expressed in the following form:

$$\begin{aligned}\frac{\partial^2}{\partial t^2}x(t,\nu) \in{}& \frac{\partial^2}{\partial \nu^2}x(t,\nu) + \varpi(t)x(t,\nu) + \int_0^t a(t-s)\frac{\partial^2 x(s,\nu)}{\partial y^2}\,\mathrm{d}s \\ &+ \beta\frac{\partial}{\partial t}x(t,\nu) + \int_0^t \kappa(s)\frac{\partial}{\partial t}x(s,\nu)\,\mathrm{d}s + \hat{\mu}(t,\nu) \\ &+ \int_{-\infty}^t \varpi_1(s-t)x\big(s-\rho_1(t)\rho_2(\|x(t)\|),\nu\big)\,\mathrm{d}s, \qquad (13_1)\end{aligned}$$

$$x(t,0) = x(t,\pi) = 0, \quad t \in [0,b],$$

$$x(\theta,\nu) = \varphi(\theta,\nu), \quad \frac{\partial}{\partial t}x(0,\nu) = \eta(\nu), \quad \theta \in (-\infty,0],\ x \in [0,\pi],$$

$$\Delta x(t_i,\nu) = \int_{-\infty}^{t_i} \nu(\mathfrak{d} - t_i)x(\mathfrak{d},\nu)\,\mathrm{d}\mathfrak{d}, \quad i = 1,2,\dots,n, \tag{13_2}$$

$$\Delta x'(t_i,\nu) = \int_{-\infty}^{t_i} \tilde{\nu}(\mathfrak{d} - t_i)x(\mathfrak{d},\nu)\,\mathrm{d}\mathfrak{d}, \quad i = 1,2,\dots,n.$$

Here $\varpi, a : [0,b] \to \mathbb{R}$, $\varpi_1 : (-\infty,0] \to \mathbb{R}$, $\rho_1 : [0,b] \to [0,\infty)$, and $\rho_2 : [0,\infty) \to [0,\infty)$ are continuous functions. The initial data $\varphi : (-\infty,0] \times [0,\pi] \to \mathbb{R}$, $\eta : [0,\pi] \to \mathbb{R}$ are appropriately defined. Furthermore, $0 < t_1 < t_2 < \dots < b$, the functions β and κ are as defined in (13). The impulse operators $\mathcal{J}_i^1, \mathcal{J}_i^2 : \mathcal{B} \to \mathcal{Y}$, $i = 1,2,\dots,n$, are bounded linear operators satisfying $\|\mathcal{J}_i^1(\xi)\| \leqslant c_i^1$ and $\|\mathcal{J}_i^2(\xi)\| \leqslant c_i^2$ for $i = 1,2,\dots,n$.

We formulate the system described in (13) within the Hilbert space framework $\mathcal{Y} = U = L^2([0,\pi])$, equipped with the standard inner product $\langle\cdot,\cdot\rangle$. In this setting, we take the functions $\varphi(\theta,\cdot)$ and $\eta(\cdot)$ to belong to $\mathcal{Y}$. Furthermore, we assume that $\mathcal{B}$ serves as an appropriate phase space for functions taking values in $\mathcal{Y}$, enabling a rigorous treatment of the system dynamics. This indicates that the mapping $\rho : [0,b] \times \mathcal{B} \to [0,\infty)$ defined by $\rho(t,\xi) = t - \rho_1(t)\rho(\|\xi(0)\|)$ is continuous and satisfies the inequality $\rho(t,\xi) \leqslant t$ for all $0 \leqslant t \leqslant b$. Furthermore, let $\varphi \in \mathcal{B}$ be such that for each $s \in (-\infty,0]$, $\varphi_s \in \mathcal{B}$, and the mapping $(-\infty,0] \to \mathcal{B}$, $s \mapsto \varphi_s$, is continuous. Additionally, there exists a nonnegative function $J^{\varphi}(s)$ such that $\|\varphi_s\| \leqslant J^{\varphi}(s)\|\varphi\|_{\mathcal{B}}$, $s \in (-\infty,0]$. Consequently, hypothesis (H_1) is satisfied. Accordingly, we define:

$$F(t,\xi) = \int_{-\infty}^{0} \varpi_1(\theta)\xi(\theta)\,\mathrm{d}\theta, \quad \xi \in \mathcal{B}.$$

We consider the mapping $F(t,\xi)$ to be well defined, and assume that F is a bounded linear operator. Consequently, there exists a constant $k > 0$ such that

$$\big\|F(t,\xi)\big\| \leqslant k\|\xi\|_{\mathcal{B}}.$$

Define the operator A_0 by $A_0x(s) = x''(z)$, where the domain of A_0 consists of all functions x for which the second derivative exists and satisfies the specified boundary conditions, that is,

$$D(A_0) = \big\{x \in L^2\big([0,\pi]\big)\colon x(0) = x(\pi) = 0\big\}.$$

Consequently, A_0 serves as the infinitesimal generator of cosine operator function $(C_0(t))_{t\in\mathbb{R}}$ on $\mathcal{Y}$, which is naturally associated withe the sine operator function $(S_0(t))_{t\in\mathbb{R}}$.

Moreover, the operator A_0 has a discrete spectrum composed of eigenvalues $-n^2$, $n \in \mathbb{N}$, with corresponding eigenfunctions

$$w_n(z) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{\mathrm{i}nz}, \quad n \in \mathbb{N}.$$

The collection w_n, $n \in \mathbb{N}$, forms an orthonormal basis of the space $\mathcal{Y}$. With respect to this basis, any element of $x \in \mathcal{Y}$ admits the representation

$$A_0 x = \sum_{n=1}^{\infty} -n^2 \langle x, w_n \rangle w_n, \quad x \in D(A_0).$$

The cosine operator function $(C_0(t))_{t \in \mathbb{R}}$ is defined by

$$C_0(t)x = \sum_{n=1}^{\infty} \cos(nt) \langle x, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

Moreover, the sine operator function $(S_0(t))_{t \in \mathbb{R}}$ is defined as follows:

$$S_0(t)x = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle x, w_n \rangle w_n, \quad t \in \mathbb{R}.$$

It follows directly from these representations that the norm of $C_0(t)$ satisfies $\|C_0(t)\| \leqslant 1$, and moreover, the operator $S_0(t)$ is compact for every $t \in \mathbb{R}$.

Look at the operator described by $A(t)x = A_0 x + a(t)x$ on $D(A)$. It is evident that $A(t)$ is a closed linear operator. Consequently, $A(t)$ generates an evolution family $S(t,s)_{0 \leqslant s \leqslant t \leqslant b}$, where each operator $S(t,s)$ is compact for all $0 \leqslant s \leqslant t \leqslant b$.

We introduce the operator $\mathcal{E}(t,s) = a(t-s)A_0$ for $0 \leqslant s \leqslant t \leqslant b$, defined on $D(A)$. With these definitions, it becomes evident that system (13) can be equivalently expressed in the abstract framework (1)–(3). Moreover, one can readily verify that conditions (Q_1)–(Q_3) outlined in Section 2 are satisfied. Consequently, there exists a resolvent operator $\mathcal{V}(t,s)_{0 \leqslant s \leqslant t \leqslant b}$ associated with (13). Furthermore, in light of (8), each operator $\mathcal{V}(t,s)$ is compact.

Let $\Gamma_0^b = \int_0^b \mathcal{V}(b,s)BB^*\mathcal{V}^*(b,s)\,\mathrm{d}s$, where the condition $S^*(b,s)x = 0$, $t \leqslant s \leqslant b$, necessarily implies that $x = 0$. Define the control operator B by $(Bv)(t)(\nu) = \hat{\mu}(t,\nu)$, and let the mappings $F : [0,b] \times \mathcal{B} \to \mathcal{Y}$, $\mathcal{D} : \mathcal{Y} \to \mathcal{Y}$, and $\rho : [0,b] \times \mathcal{B} \to \mathbb{R}$ be specified as follows:

$$F(t,\psi)(\nu) = \int_{-\infty}^{0} \varpi_1(s)\psi(s,\nu)\,\mathrm{d}s,$$

$$\mathcal{D}\psi(\nu) = \beta\psi(t,\nu) + \int_0^{\pi} \kappa(s)\psi(t,s)\,\mathrm{d}s,$$

$$\rho(t,\psi) = t - \rho_1(t)\rho_2\bigl(\|\psi(0)\|\bigr),$$

$$\mathcal{J}_i^1(\psi)(\nu) = \int_{-\infty}^{0} \nu(\mathfrak{d})\psi(\mathfrak{d},\nu)\,\mathrm{d}\mathfrak{d}, \quad i = 1,2,\ldots,n,$$

$$\mathcal{J}_i^2(\psi)(\nu) = \int_{-\infty}^{0} \tilde{\nu}(\mathfrak{d})\psi(\mathfrak{d},\nu)\,\mathrm{d}\mathfrak{d}, \quad i = 1,2,\ldots,n.$$

Assuming that the functions fulfill the required hypotheses and taking the evolution operator $A(t)$ with $B = I$, system (13) can be reformulated as an abstract second-order semilinear system. In this form, it is represented by (1)–(3) in the space $\mathcal{Y}$. Since all the conditions of Theorem 2 are satisfied, it follows that system (13) is approximately controllable on the interval $[0, b]$.

5 Conclusion

This paper investigates the controllability of damped impulsive second-order integro-differential nonautonomous evolution systems with state-dependent delay. First, sufficient conditions for the existence of mild solutions to the proposed system are established using Schauder's fixed-point theorem and essential properties of the resolvent operator associated with second-order integro-differential systems. Furthermore, approximate controllability of the system is demonstrated under a new set of sufficient conditions. To illustrate the applicability of the theoretical results, a representative example is presented. Future research may extend this framework to mixed Volterra–Fredholm-type stochastic differential systems.

Author contributions. The authors (R.S. and V.V.) have contributed as follows: methodology, R.S.; formal analysis, R.S. and V.V.; validation, V.V.; writing – original draft preparation, R.S. and V.V.; writing – review & editing, R.S. and V.V. Both authors have read and approved the published version of the manuscript.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. K. Aissani, M. Benchohra, Impulsive fractional differential inclusions with state-dependent delay, *Math. Morav.*, **23**(2):97–113, 2019, https://doi.org/10.5937/MatMor1902097A.

2. M.M. Arjunan, N.Y. Nadaf, Existence and controllability results for damped second-order impulsive functional differential systems with state-dependent delay, *Opusc. Math.*, **34**(3):503–522, 2014, https://doi.org/10.7494/OpMath.2014.34.3.503.

3. G. Arthi, K. Balachandran, Controllability of damped second-order impulsive neutral functional differential systems with infinite delay, *J. Optim. Theory Appl.*, **152**:799–813, 2012, https://doi.org/10.1007/s10957-011-9926-z.

4. D. Bainov, P. Simeonov, *Impulsive Differential Equations: Periodic Solutions and Applications*, Routledge, New York, 2017, https://doi.org/10.1201/9780203751206.

5. M. Benchohra, S.K. Ntouyas, Controllability of second-order differential inclusions in Banach spaces with nonlocal conditions, *J. Optim. Theory Appl.*, **107**(3):559–571, 2000, https://doi.org/10.1023/A:1026447232030.

6. H.O. Fattorini, *Second-Order Linear Differential Equations in Banach Spaces*, North-Holland Math. Stud., Vol. 108, Elsevier, Amsterdam, 2011, https://doi.org/10.1016/S0304-0208(08)73159-1.

7. H.R. Henríquez, Existence of solutions of non-autonomous second order functional differential equations with infinite delay, *Nonlinear Anal., Theory Methods Appl.*, **74**:3333–3352, 2011, https://doi.org/10.1016/j.na.2011.02.010.

8. H.R. Henríquez, E. Hernández, Approximate controllability of second-order distributed implicit functional systems, *Nonlinear Anal., Theory Methods Appl.*, **70**(2):1023–1039, 2009, https://doi.org/10.1016/j.na.2008.01.029.

9. H.R. Henríquez, J.C. Pozo, Existence of solutions of abstract non-autonomous second order integro-differential equations, *Bound. Value Probl.*, **2016**:168, 2016, https://doi.org/10.1186/s13661-016-0675-7.

10. E. Hernández, K. Balachandran, N. Annapoorani, Existence results for a damped second-order abstract functional differential equation with impulses, *Math. Comput. Modellig*, **50**:1583–1594, 2009, https://doi.org/10.1016/j.mcm.2009.09.007.

11. E. Hernández, H.R. Henríquez, M.A. McKibben, Existence results for abstract impulsive second-order neutral functional differential equations, *Nonlinear Anal., Theory Methods Appl.*, **70**(7):2736–2751, 2009, https://doi.org/10.1016/j.na.2008.03.062.

12. E. Hernández, A. Prokopczyk, L. Ladeira, A note on partial functional differential equations with state-dependent delay, *Nonlinear Anal., Real World Appl.*, **7**(4):510–519, 2006, https://doi.org/10.1016/j.nonrwa.2005.03.014.

13. Y. Hino, S. Murakami, T. Naito, *Functional Differential Equations with Infinite Delay*, Lect. Notes Math., Vol. 1473, Springer, Berlin, Heidelberg, 1991, https://doi.org/10.1007/BFb0084432.

14. K. Karthikeyan, D. Tamizharasan, J.J. Nieto, K.S. Nisar, Controllability of second-order differential equations with state-dependent delay, *IMA J. Math. Control Inf.*, **38**(4):1072–1083, 2021, https://doi.org/10.1093/imamci/dnab027.

15. V. Lakshmikantham, D.D. Bainov, P.S. Simeonov, *Theory of Impulsive Differential Equations*, Ser. Mod. Appl. Math., Vol. 6, World Scientific, Singapore, 1989, https://doi.org/10.1142/0906.

16. N.I. Mahmudov, A. Denker, On controllability of linear stochastic systems, *Int. J. Control*, **73**(2):144–151, 2000, https://doi.org/10.1080/002071700219849.

17. S. Rezapour, H.R. Henríquez, V. Vijayakumar, K.S. Nisar, A. Shukla, A note on existence of mild solutions for second-order neutral integro-differential evolution equations with state-dependent delay, *Fractal Fract.*, **5**(3):126, 2021, https://doi.org/10.3390/fractalfract5030126.

18. R. Sakthivel, E.R. Anandhi, Approximate controllability of impulsive differential equations with state-dependent delay, *Int. J. Control*, **83**(2):387–393, 2010, https://doi.org/10.1080/00207170903171348.
19. R. Sakthivel, E.R. Anandhi, N.I. Mahmudov, Approximate controllability of second-order systems with state-dependent delay, *Numer. Funct. Anal. Optim.*, **29**(11-12):1347–1362, 2008, https://doi.org/10.1080/01630560802580901.
20. R. Sakthivel, Y. Ren, N.I. Mahmudov, On the approximate controllability of fractional differential systems, *Comput. Math. Appl.*, **62**(3):1451–1459, 2011, https://doi.org/10.1016/j.camwa.2011.04.040.
21. R. Sasikumar, V. Vijayakumar, Discussion on the approximate controllability of second-order neutral differential inclusions with damping, *Qual. Theory Dyn. Syst.*, **23**(1):284, 2024, https://doi.org/10.1007/s12346-024-01150-5.
22. R. Sasikumar, V. Vijayakumar, Approximate controllability of damped second-order neutral differential equations with state-dependent delay, *Int. J. Control*, **98**(6):1326–1334, 2025, https://doi.org/10.1080/00207179.2024.2395316.
23. R. Sasikumar, V. Vijayakumar, A class of second-order damped neutral stochastic differential control systems with state-dependent delay and impulses, *Complex Anal. Oper. Theory*, **19**:1–27, 2025, https://doi.org/10.1007/s11785-025-01794-4.
24. S. Singh, S. Arora, M.T. Mohan, J. Dabas, Approximate controllability of second-order impulsive systems with state-dependent delay in Banach spaces, *Evol. Equ. Control Theory*, **11**(1):1–27, 2022, https://doi.org/10.3934/eect.2020103.
25. V. Singh, D.N. Pandey, Controllability of fractional impulsive quasilinear differential systems with state dependent delay, *Int. J. Dyn. Control*, **7**:313–325, 2019, https://doi.org/10.1007/s40435-018-0425-z.
26. V. Singh, D.N. Pandey, Controllability of second-order Sobolev-type impulsive delay differential systems, *Math. Methods Appl. Sci.*, **42**(5):1377–1388, 2019, https://doi.org/10.1002/mma.5427.
27. V. Singh, D.N. Pandey, Exact controllability of multi-term time-fractional differential system with sequencing techniques, *Indian J. Pure Appl. Math.*, **51**:105–120, 2020, https://doi.org/10.1007/s13226-020-0387-7.
28. C.C. Travis, G.F. Webb, Cosine families and abstract nonlinear second-order differential equations, *Acta Math. Acad. Sci. Hung.*, **32**:76–96, 1978, https://doi.org/10.1007/BF01902205.
29. V. Vijayakumar, A. Shukla, K.S. Nisar, W. Jamshed, S. Rezapour, A note on the approximate controllability of second-order integro-differential evolution control systems via resolvent operators, *Adv. Difference Equ.*, **2021**:484, 2021, https://doi.org/10.1186/s13662-021-03639-8.
30. T. Yang, *Impulsive Control Theory*, Lect. Notes Control Inf. Sci., Vol. 272, Springer, Berlin, Heidelberg, 2001, https://doi.org/10.1007/3-540-47710-1.