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Fixed point results for four-point Kannan mappings with applications to fractional BVPs and Diophantine equations

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Abstract. In this paper, we introduce and study Kannan-type aggregate pairwise distance mappings defined on four points, formulated through mapping conditions involving the sum of all pairwise distances of a quadrilateral configuration. We analyze their structural properties and clarify their connections with classical Kannan mappings, generalized Kannan-type mappings, as well as with mappings involving the total pairwise distance of a quadrilateral. We further extend the theory by introducing functional generalizations, namely the G-Kannan-type and $\mathcal{B}$-Kannan-type aggregate pairwise distance mappings on four points, in which the constant parameter is replaced by appropriate control functions. Under the assumptions of asymptotic regularity or approximating fixed point sequence, with or without weaker continuity hypotheses, we obtain functional extensions that improve the admissible range of the associated parameters. Finally, applications are provided to the existence of solutions for fractional differential equations with boundary conditions and to a nonlinear Diophantine equation.

Keywords: fixed point theorem, Kannan mappings, asymptotic regularity, fractional differential equations, Diophantine equation.

1 Introduction

The Banach contraction principle occupies a central position in fixed point theory and is widely regarded as one of its most fundamental and pivotal results. Its importance lies not only in guaranteeing the existence and uniqueness of fixed points for a broad class of self-mappings on complete metric spaces but also in providing a constructive method for approximating those fixed points. This combination of theoretical depth and practical

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utility has paved the way for its extensive use across numerous branches of mathematics and the applied sciences (see [18, 33] and the references therein).

In 1968, Kannan [20] proved a fixed point theorem for a class of mappings that are neither contractions nor contractive. Later, Subrahmanyam [32] established that Kannan's theorem uniquely characterizes metric completeness, a property not shared by the Banach contraction principle. Specifically, a metric space X has the metric completeness property if and only if every Kannan mapping on X has a fixed point.

A mapping $T : X \to X$ defined on a metric space (X, d) is said to be a *Kannan mapping* if it satisfies the inequality

$$d(Tx, Ty) \leqslant \lambda\bigl(d(x, Tx) + d(y, Ty)\bigr), \quad x, y \in X,\ 0 \leqslant \lambda < 0.5. \tag{1}$$

Every Kannan mapping on a complete metric space has a unique fixed point [20].

Building on the foundational work of Banach and Kannan, researchers have significantly advanced fixed point theory by establishing results under diverse contractive conditions (see [7, 30] and references therein). These developments have led to broader generalizations of classical theorems, achieved by relaxing contractive conditions, integrating additional distances, modifying topological structures, and redefining metric spaces to enhance their applicability. The extension to multivalued mappings has further expanded the scope, enabling the exploration of fixed points in varied contexts, including common fixed points and best proximity points, while ensuring validity in generalized metric spaces. These innovations not only unify and extend fundamental principles but also reinforce the relevance of fixed point theory in numerous scientific and engineering applications, as highlighted in [1, 3, 5, 11, 14–17, 19, 21, 22, 24, 29, 31, 33].

In 2023, more than a century after the Banach contraction principle, Petrov [25] introduced a new class of mappings contracting the perimeters of triangles, defined for three pairwise distinct points, together with an additional restriction excluding periodic points of prime period 2. This development effectively extended the Banach contraction principle to a three-point framework. In 2025, Petrov [26] further advanced this line of research by introducing mappings that contract the total pairwise distances among k points in metric spaces, accompanied by an existence theorem for their periodic points, marking a significant evolution in the study of contraction mappings.

In this paper, $|X|$ denotes the cardinality of a set X. The symbols $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{Q}^c$, $\mathbb{R}$, and $\mathbb{R}^+$ denote the sets of natural numbers, rational numbers, irrational numbers (i.e., $\mathbb{R} \setminus \mathbb{Q}$), real numbers, and positive real numbers, respectively. Let $T : X \to X$. A point $x \in X$ is called a fixed point of T if $T(x) = x$; it is called a periodic point of T if there exists $n \in \mathbb{N}$ such that $T^n(x) = x$; and it is called a periodic point of prime order p if $p \in \mathbb{N}$ is prime, $T^p(x) = x$, and $T^k(x) \neq x$ for every $1 \leqslant k < p$ [12].

Definition 1. (See [26].) Let $k \geqslant 3$, $k \in \mathbb{N}$, and let (X, d) be a metric space with $|X| \geqslant k$. A mapping $T : X \to X$ is called a *mapping contracting the total pairwise distances between k points* if there exists $\alpha \in [0, 1)$ such that

$$\sum_{1 \leqslant i < j \leqslant k} d(Tx_i, Tx_j) \leqslant \alpha \sum_{1 \leqslant i < j \leqslant k} d(x_i, x_j) \tag{2}$$

holds for all k pairwise distinct points $x_1, x_2, \ldots, x_k \in X$.

In 2023, Petrov and Bisht [27] introduced generalized Kannan-type mappings, a new class of three-point analogues of Kannan mappings in metric spaces.

Definition 2. Let (X,d) be a metric space with $|X| \geqslant 3$. A mapping $T : X \to X$ is called a *generalized Kannan-type mapping* if there exists a constant $\lambda \in [0, 2/3)$ such that

$$\begin{aligned} &d(Tx_1, Tx_2) + d(Tx_2, Tx_3) + d(Tx_3, Tx_1) \\ &\quad \leqslant \lambda\big(d(x_1, Tx_1) + d(x_2, Tx_2) + d(x_3, Tx_3)\big) \end{aligned}$$

holds for all three pairwise distinct points $x_1, x_2, x_3 \in X$.

More recently, Banerjee et al. [4] introduced a four-point analogue of the Kannan-type contraction as follows:

Definition 3. Let (X,d) be a metric space with $|X| \geqslant 4$. A mapping $T : X \to X$ is called a *Kannan-type perimetric contraction on quadrilaterals* if there exists a constant $\delta \in [0, 1/2)$ such that

$$\begin{aligned} &d(Tx_1, Tx_2) + d(Tx_2, Tx_3) + d(Tx_3, Tx_4) + d(Tx_4, Tx_1) \\ &\quad \leqslant \delta\big(d(x_1, Tx_1) + d(x_2, Tx_2) + d(x_3, Tx_3) + d(x_4, Tx_4)\big) \end{aligned} \tag{3}$$

holds for all four pairwise distinct points $x_1, x_2, x_3, x_4 \in X$.

Remark 1. It is important to note that condition (3) focuses on the perimeter of the quadrilateral formed by the images of the four points.

Various other works employing multipoint Banach contractions can be found in [8, 9, 19], including those developed within the framework of semimetric spaces based on the triangle functions introduced by Bessenyei and Páles [6].

In this paper, we introduce the notion of a Kannan-type aggregate pairwise distance mapping on four points, which generalizes Kannan-type perimetric contractions on quadrilaterals by considering the entire structure of the quadrilateral, including its internal connections and diagonals as follows:

Definition 4. Let (X,d) be a metric space with $|X| \geqslant 4$. We say that a mapping $T : X \to X$ is a *Kannan-type aggregate pairwise distance mapping on four points* (or a *Kannan-type aggregate pairwise distance mapping on a quadrilateral*) if there exists a constant $\lambda \in [0, 3/4)$ such that

$$\begin{aligned} &d(Tx_1, Tx_2) + d(Tx_1, Tx_3) + d(Tx_1, Tx_4) \\ &\quad + d(Tx_2, Tx_3) + d(Tx_2, Tx_4) + d(Tx_3, Tx_4) \\ &\quad \leqslant \lambda\big(d(x_1, Tx_1) + d(x_2, Tx_2) + d(x_3, Tx_3) + d(x_4, Tx_4)\big) \end{aligned} \tag{4}$$

holds for all four pairwise distinct points $x_1, x_2, x_3, x_4 \in X$.

Remark 2. It is pertinent to note that for four points x_1, x_2, x_3, and x_4 in a metric space, there are exactly $\binom{4}{2} = 6$ pairwise distances. Geometrically, these correspond to

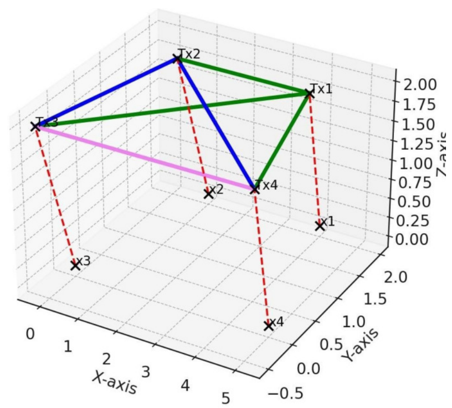

Figure 1. Geometric depiction of a Kannan-type aggregate pairwise distance mapping on four points.

the four boundary edges and the two diagonals of the quadrilateral determined by the points. Thus, condition (4) captures the entire geometric configuration of the four points, rather than only the boundary structure considered in (3) (see Fig. 1).

Example 1. Let (X, d) be a metric space, where $X = \{x_1, x_2, x_3, x_4\}$ with metric

$$d(x,y) = \begin{cases} 0, & x = y, \\ 1, & \{x,y\} = \{x_1, x_2\}, \\ 4 & \text{otherwise}, \end{cases}$$

and define $T : X \to X$ by $T(x_1) = x_1$, $T(x_2) = x_2$, $T(x_3) = x_1$, and $T(x_4) = x_2$.

Since $\sum_{1 \leqslant i<j \leqslant 4} d(Tx_i, Tx_j) = 4$ and $\sum_{i=1}^{4} d(x_i, Tx_i) = 8$, the condition $4 \leqslant \lambda \cdot 8$ holds for all $\lambda \in [1/2, 3/4)$, so T is a Kannan-type aggregate pairwise distance mapping on four points for $\lambda \in [1/2, 3/4)$. On the other hand, the cyclic sum

$$d(Tx_1, Tx_2) + d(Tx_2, Tx_3) + d(Tx_3, Tx_4) + d(Tx_4, Tx_1) = 4$$

gives $4 \leqslant \delta \cdot 8$, i.e., $\delta \geqslant 1/2$. Since condition (3) requires $\delta < 1/2$, T does not satisfy the Kannan-type perimetric contraction condition on a quadrilateral.

This paper is organized as follows. Section 2 investigates the structural properties of Kannan-type aggregate pairwise distance mappings defined on four points. In particular, we examine their relationship with classical Kannan-type contractions, generalized Kannan-type mappings, and mappings involving the total pairwise distance of a quadrilateral. We also show that such a mapping may fail to be continuous at every point of its domain except at its fixed points, which constitute the only points of continuity.

Section 3 establishes the main result of the paper, Theorem 1, which provides a fixed point theorem for Kannan-type aggregate pairwise distance mappings on four points. The theorem asserts that the number of fixed points is at most three, provided that T does not admit periodic points of prime periods 2 and 3. Moreover, we prove that these mappings are continuous at their fixed points.

Section 4 extends Theorem 1 in two directions. First, we broaden the class of admissible constants in (4) by introducing functional generalizations, namely the G-Kannan-type and $\mathcal{B}$-Kannan-type variants. Under the assumptions of asymptotic regularity or approximating fixed point sequences, together with weaker forms of continuity, we establish functional extensions of the main result in Theorems 2 and 3, allowing the parameter λ to vary over $[0, \infty)$, as shown in Corollaries 2 and 3. Second, by dropping the continuity or weaker continuity assumptions and assuming only asymptotic regularity or approximating fixed point sequences, we improve the admissible range of λ in (4) from $[0, 3/4)$ to $[0, 1)$, as established in Theorem 4.

Section 5 is devoted to applications. Subsection 5.1 applies Theorem 1 to study the existence of solutions to fractional differential equations with boundary conditions, while Subsection 5.2 presents an application to study the solutions of a nonlinear Diophantine equation.

2 Properties of Kannan-type aggregate pairwise distance mappings on four points

The following proposition shows that the standard Kannan class with $0 \leqslant \lambda < 1/4$ is contained within the broader framework of Kannan-type aggregate pairwise distance mappings on four points.

Proposition 1. *A Kannan mapping with parameter $\lambda \in [0, 1/4)$ is a Kannan-type aggregate pairwise distance mapping on four points.*

Proof. Let (X, d) be a metric space with $|X| \geqslant 4$, and let $T : X \to X$ be a Kannan mapping with parameter $\lambda \in [0, 1/4)$. Consider four pairwise distinct points $x_1, x_2, x_3, x_4 \in X$. Applying (1) to all distinct pairs x_i, x_j ($1 \leqslant i < j \leqslant 4$), we obtain

$$d(Tx_i, Tx_j) \leqslant \lambda\bigl(d(x_i, Tx_i) + d(x_j, Tx_j)\bigr), \quad 1 \leqslant i < j \leqslant 4.$$

Summing over all such pairs, we get

$$\sum_{1\leqslant i<j\leqslant 4} d(Tx_i, Tx_j) \leqslant \lambda \sum_{1\leqslant i<j\leqslant 4} \bigl(d(x_i, Tx_i) + d(x_j, Tx_j)\bigr). \tag{5}$$

Each $d(x_k, Tx_k)$ appears in exactly $\binom{3}{1} = 3$ of the six pairs, so

$$\sum_{1\leqslant i<j\leqslant 4} \bigl(d(x_i, Tx_i) + d(x_j, Tx_j)\bigr) = 3\sum_{i=1}^{4} d(x_i, Tx_i).$$

Thus, it follows from (5) that

$$\sum_{1\leqslant i<j\leqslant 4} d(Tx_i, Tx_j) \leqslant 3\lambda\sum_{i=1}^{4} d(x_i, Tx_i).$$

Since $\lambda \in [0, 1/4)$, we have $3\lambda \in [0, 3/4)$. Hence, T satisfies condition (4) of Definition 4, and therefore, it is a Kannan-type aggregate pairwise distance mapping on four points. □

Proposition 2. *Let (X, d) be a metric space, and let $T : X \to X$ be a Kannan-type aggregate pairwise distance mapping on four points with some $\lambda \in [0, 3/4)$. If x is an accumulation point of X and T is continuous at x, then the inequality*

$$d(Tx, Ty) \leqslant \lambda\left(d(x, Tx) + \frac{d(y, Ty)}{3}\right) \tag{6}$$

holds for all points $y \in X$.

Proof. Let $x \in X$ be an accumulation point, and let $y \in X$. If $y = x$, then (6) holds trivially since both sides equal zero. Now assume that $y \neq x$. Since x is an accumulation point, there exists a sequence $\{z_n\} \in X$ with $z_n \to x$, $z_n \neq x$, $z_n \neq y$, and all z_n distinct. Hence, x, y, z_n, and z_{n+1} are pairwise distinct for all $n \in \mathbb{N}$. Applying condition (4) to these four points gives

$$\begin{aligned} &d(Tx, Ty) + d(Tx, Tz_n) + d(Tx, Tz_{n+1}) \\ &\quad + d(Ty, Tz_n) + d(Ty, Tz_{n+1}) + d(Tz_n, Tz_{n+1}) \\ &\quad \leqslant \lambda\big(d(x, Tx) + d(y, Ty) + d(z_n, Tz_n) + d(z_{n+1}, Tz_{n+1})\big). \end{aligned} \tag{7}$$

Since $z_n \to x$ and T is continuous at x, it follows that $Tz_n \to Tx$ and $Tz_{n+1} \to Tx$. By the joint continuity of d, we also have $d(z_n, Tz_n) \to d(x, Tx)$ and $d(z_{n+1}, Tz_{n+1}) \to d(x, Tx)$. Taking $n \to \infty$ in (7) yields (6). □

Corollary 1. *Let (X, d) be a metric space, and let $T : X \to X$ be a continuous Kannan-type aggregate pairwise distance mapping on four points. If all points of X are accumulation points, then T is a Kannan mapping.*

Proof. Applying Proposition 2 to the pairs (x, y) and (y, x) and adding the resulting inequalities gives

$$2\, d(Tx, Ty) \leqslant \frac{4\lambda}{3}\big(d(x, Tx) + d(y, Ty)\big).$$

Dividing by 2, we obtain $d(Tx, Ty) \leqslant (2\lambda/3)(d(x, Tx) + d(y, Ty))$, where $2\lambda/3 \in [0, 1/2)$ since $\lambda \in [0, 3/4)$. Thus, T is a Kannan mapping. □

Remark 3. The assumption that x is an accumulation point in Proposition 2 is essential. It guarantees the existence of a sequence of distinct points converging to x, which allows condition (4) to be applied prior to taking limits. If x were isolated, such a sequence could not exist, and the conclusions of Proposition 2 and Corollary 1 may fail.

Proposition 3. *Let (X, d) be a metric space with $|X| \geqslant 4$, and let $T : X \to X$ be a mapping contracting total pairwise distance on four points with a constant $0 \leqslant \alpha < 1/6$. Then T is a Kannan-type aggregate pairwise distance mapping on four points with respect to the metric d.*

Proof. Let $x_1, x_2, x_3, x_4 \in X$ be pairwise distinct. By (2), we get

$$\sum_{1 \leqslant i<j \leqslant 4} d(Tx_i, Tx_j) \leqslant \alpha \sum_{1 \leqslant i<j \leqslant 4} d(x_i, x_j).$$

Applying the triangle inequality $d(x_i, x_j) \leqslant d(x_i, Tx_i) + d(Tx_i, Tx_j) + d(Tx_j, x_j)$ to each of the six pairs and summing, each $d(x_k, Tx_k)$ appears in exactly three pairs, so

$$\sum_{1 \leqslant i<j \leqslant 4} d(x_i, x_j) \leqslant 3 \sum_{i=1}^{4} d(x_i, Tx_i) + 2 \sum_{1 \leqslant i<j \leqslant 4} d(Tx_i, Tx_j).$$

Substituting and collecting terms, we get

$$(1-2\alpha) \sum_{1 \leqslant i<j \leqslant 4} d(Tx_i, Tx_j) \leqslant 3\alpha \sum_{i=1}^{4} d(x_i, Tx_i).$$

Since $\alpha < 1/6$ implies $1 - 2\alpha > 0$, dividing gives

$$\sum_{1 \leqslant i<j \leqslant 4} d(Tx_i, Tx_j) \leqslant \frac{3\alpha}{1-2\alpha} \sum_{i=1}^{4} d(x_i, Tx_i).$$

Setting $\lambda = 3\alpha/(1-2\alpha) \in [0, 3/4)$, T is a Kannan-type aggregate pairwise distance mapping on four points. □

We now present an example demonstrating that a Kannan-type aggregate pairwise distance mapping on four points can be discontinuous at every point of its domain except at its fixed point, which is its only point of continuity.

Example 2. Let $X = [0, 2]$ be equipped with the usual Euclidean metric d, and define a mapping $T : X \to X$ by

$$T(x) = \begin{cases} \frac{x}{6} & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{Q}^c. \end{cases}$$

Then T satisfies the Kannan mapping, that is,

$$d(Tx, Ty) \leqslant \frac{1}{5}\big(d(x, Tx) + d(y, Ty)\big)$$

for all $x, y \in X$. Therefore, by Proposition 1, T qualifies as a Kannan-type aggregate pairwise distance mapping on four points. Moreover, T is discontinuous at every point of X except at $x = 0$, which is the unique fixed point of T.

3 Main results

The following theorem constitutes the main result of this work.

Theorem 1. *Let (X, d), $|X| \geqslant 4$, be a complete metric space, and let the mapping $T : X \to X$ be a Kannan-type aggregate pairwise distance mapping on four points. If T does not have periodic points of prime period 2 and 3, then T has at least one fixed point in X. Furthermore, T has at most three fixed points.*

Proof. Let $x_0 \in X$ be chosen arbitrarily. Define the sequence $\{x_n\}$ recursively by

$$x_{n+1} = Tx_n, \quad \text{for all } n \geqslant 0.$$

If x_n is a fixed point of T for some n, then we are done.

Now assume that x_n is not a fixed point of T for any $n \geqslant 0$. Since the sequence is defined by $x_{n+1} = Tx_n$, this implies $x_n \neq Tx_n = x_{n+1}$ for all n. If $x_n = x_{n+2}$ for some n, then $x_{n+1} = T^2 x_{n+1}$; since x_{n+1} is not a fixed point, it has prime period 2, contradicting the hypothesis. Similarly, $x_n = x_{n+3}$ implies $x_n = T^3 x_n$; since x_n is not a fixed point, it has prime period 3, which is again a contradiction. Hence, for every n, the four consecutive terms x_n, x_{n+1}, x_{n+2}, and x_{n+3} are pairwise distinct.

Applying (4) to x_n, x_{n+1}, x_{n+2}, and x_{n+3} and substituting $Tx_k = x_{k+1}$, we obtain

$$\begin{aligned} &d(x_{n+1}, x_{n+2}) + d(x_{n+1}, x_{n+3}) + d(x_{n+1}, x_{n+4}) \\ &\quad + d(x_{n+2}, x_{n+3}) + d(x_{n+2}, x_{n+4}) + d(x_{n+3}, x_{n+4}) \\ &\quad \leqslant \lambda\bigl(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+4})\bigr). \end{aligned}$$

Rearranging terms, we obtain

$$\begin{aligned} &(1-\lambda) d(x_{n+3}, x_{n+4}) \\ &\quad \leqslant \lambda\bigl(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3})\bigr) \\ &\qquad - \bigl(d(x_{n+1}, x_{n+3}) + d(x_{n+1}, x_{n+4})\bigr) - \bigl(d(x_{n+2}, x_{n+3}) + d(x_{n+2}, x_{n+4})\bigr) \\ &\qquad - d(x_{n+1}, x_{n+2}). \end{aligned}$$

By the triangle inequality applied to the triples $x_{n+1}, x_{n+3}, x_{n+4}$ and $x_{n+2}, x_{n+3}, x_{n+4}$,

$$-d(x_{n+1}, x_{n+3}) - d(x_{n+1}, x_{n+4}) \leqslant -d(x_{n+3}, x_{n+4})$$

and

$$-d(x_{n+2}, x_{n+3}) - d(x_{n+2}, x_{n+4}) \leqslant -d(x_{n+3}, x_{n+4}).$$

Omitting the negative term $-d(x_{n+1}$ and $x_{n+2})$, we obtain

$$(3-\lambda) d(x_{n+3}, x_{n+4}) \leqslant \lambda\bigl(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3})\bigr).$$

Therefore,

$$d(x_{n+3}, x_{n+4}) \leqslant \frac{3\lambda}{3-\lambda} \max\bigl(d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_{n+2}, x_{n+3})\bigr).$$

Let $\delta = 3\lambda/(3-\lambda) \in [0, 1)$ and $a_n = d(x_n, x_{n+1})$ with $a = \max\{a_1, a_2, a_3\}$. Since $a_{n+3} \leqslant \delta \max\{a_n, a_{n+1}, a_{n+2}\}$, an induction gives $a_n \leqslant a\, \delta^{\lfloor (n-1)/3 \rfloor}$ for all $n \geqslant 1$.

Therefore, for any $p = 1, 2, \dots,$

$$d(x_n, x_{n+p}) \leqslant \sum_{k=0}^{p-1} a_{n+k} \leqslant \frac{a\,\delta^{\lfloor (n-1)/3 \rfloor}}{1-\delta^{1/3}} \to 0 \quad \text{as } n \to \infty.$$

Hence $\{x_n\}$ is a Cauchy sequence in X. By completeness, $x_n \to x^*$ for some $x^* \in X$.

Since any four consecutive terms are pairwise distinct, if $x^* = x_k$ for some smallest k, the sequence would be cyclic and hence not Cauchy. Thus, x_{n-1}, x_n, x_{n+1}, and x^* are pairwise distinct for all sufficiently large n. We have

$$\begin{aligned} d(x^*, Tx^*) &\leqslant d(x^*, x_n) + d(x_n, Tx^*) \\ &= d(x^*, x_n) + d(Tx_{n-1}, Tx^*) \\ &\leqslant d(x^*, x_n) + d(Tx^*, Tx_{n-1}) + d(Tx^*, Tx_{n-2}) + d(Tx^*, Tx_n) \\ &\quad + d(Tx_{n-1}, Tx_{n-2}) + d(Tx_{n-1}, Tx_n) + d(Tx_{n-2}, Tx_n) \\ &\leqslant d(x^*, x_n) + \lambda\bigl(d(x^*, Tx^*) + d(x_{n-1}, Tx_{n-1}) \\ &\quad + d(x_{n-2}, Tx_{n-2}) + d(x_n, Tx_n)\bigr). \end{aligned}$$

This implies that

$$\begin{aligned} &(1-\lambda) d(x^*, Tx^*) \\ &\quad \leqslant d(x^*, x_n) + \lambda\bigl(d(x_{n-1}, x_n) + d(x_{n-2}, x_{n-1}) + d(x_n, x_{n+1})\bigr). \end{aligned} \tag{8}$$

Taking $n \to \infty$, we get $Tx^* = x^*$.

If there exist four pairwise distinct fixed points x, y, z, and w, substituting $Tx = x$, $Ty = y$, $Tz = z$, and $Tw = w$ into (4) yields

$$d(x,y) + d(x,z) + d(x,w) + d(y,z) + d(y,w) + d(z,w) \leqslant 0,$$

a contradiction. Hence T has at most three fixed points. □

Remark 4. Suppose that under the assumptions of Theorem 1, the mapping T has a fixed point x^*, which is the limit of the iterative sequence

$$x_0, \quad x_1 = Tx_0, \quad x_2 = Tx_1, \quad \dots$$

such that $x_n \neq x^*$ for all $n \geqslant 1$. Then x^* is the unique fixed point of T.

Assume, for contradiction, that T has another fixed point $x^{**} \neq x^*$. Clearly, $x_n \neq x^{**}$ for all $n \geqslant 1$. Hence, for each n, the points x^*, x^{**}, x_n, and x_{n+1} are pairwise distinct.

Let us denote

$$x_1 = x^*, \quad x_2 = x^{**}, \quad x_3 = x_n, \quad x_4 = x_{n+1}.$$

Consider the ratio

$$R_n = \frac{\sum_{1 \leqslant i < j \leqslant 4} d(Tx_i, Tx_j)}{\sum_{i=1}^{4} d(x_i, Tx_i)}.$$

Since x^* and x^{**} are fixed points of T, the above ratio reduces to

$$R_n = \frac{1}{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})} \bigl(d(x^*, x^{**}) + d(x^*, x_{n+1}) + d(x^*, x_{n+2}) + d(x^{**}, x_{n+1}) + d(x^{**}, x_{n+2}) + d(x_{n+1}, x_{n+2})\bigr).$$

Since $x_n \to x^*$ as $n \to \infty$, it follows that

$$d(x_n, x_{n+1}) \to 0 \quad \text{and} \quad d(x_{n+1}, x_{n+2}) \to 0 \quad \text{as } n \to \infty.$$

On the other hand, $d(x^*, x^{**}) > 0$ whenever $x^* \neq x^{**}$. Consequently,

$$R_n \to \infty \quad \text{as } n \to \infty,$$

which contradicts condition (4). Hence, $x^* = x^{**}$, and therefore x^* is the unique fixed point of T.

We now provide examples that illustrate Theorem 1.

Example 3. Let (X, d) be a metric space such that $X = \{x_1, x_2, x_3, x_4\}$ and

$$d(x_i, x_j) = 1, \quad \text{for } i \neq j,\ 1 \leqslant i, j \leqslant 3,$$
$$d(x_i, x_4) = 9, \quad \text{for } 1 \leqslant i \leqslant 3.$$

Define the mapping $T : X \to X$ as follows:

$$T(x_1) = x_1, \qquad T(x_2) = x_2, \qquad T(x_3) = x_3, \qquad T(x_4) = x_1.$$

Then T is a Kannan-type aggregate pairwise distance mapping on four points for $\lambda \in [5/9, 3/4)$. Also, T does not contain any periodic points of prime periods 2 and 3. Therefore, by Theorem 1, T has three fixed points: x_1, x_2, and x_3.

Example 4. Let $\{n_i, i \geqslant 1\}$ be the Fibonacci-type sequence defined by

$$n_1 = 1, \quad n_2 = 5, \quad n_i = 6n_{i-1} - n_{i-2} \quad \text{for } i \geqslant 3,$$

which yields $n_3 = 29$, $n_4 = 169$, $n_5 = 985$, $n_6 = 5741, \dots$.

Let $X = [1, \infty)$ be equipped with the Euclidean metric. Define $T : X \to X$ by

$$T(n_i) = n_{i-1} \quad \text{for } i \geqslant 2, \qquad T(x) = 1 \quad \text{for all other } x \in X.$$

Then T satisfies the Kannan-type aggregate pairwise distance condition with $\lambda \in [3(\sqrt{2} - 1)/2,\ 3/4)$. Moreover, T has a unique fixed point at $x = 1$, is nonlinear, and also satisfies the conditions of Theorem 1.

Example 5. Let $\{m_i,\ i \geqslant 1\}$ be the Fibonacci-type sequence defined by

$$m_1 = 0, \quad m_2 = 3, \qquad m_i = 6m_{i-1} - (m_{i-2} - 2) \quad \text{for } i \geqslant 3,$$

yielding $m_3 = 20$, $m_4 = 119$, $m_5 = 696$, $m_6 = 4059, \dots$.

Let $X = [0, \infty)$ with Euclidean metric. Define $T : X \to X$ by

$$T(m_i) = m_{i-1} \quad \text{for } i \geqslant 2, \qquad T(x) = 0 \quad \text{otherwise.}$$

Then T is a Kannan-type aggregate pairwise distance mapping with $\lambda \in [3(\sqrt{2}-1)/2, 3/4)$. Moreover, T has a unique fixed point at $x = 0$, is nonlinear, and also satisfies the conditions of Theorem 1.

We state the following proposition, which establishes the continuity of a Kannan-type aggregate pairwise distance mapping on four points at its fixed points.

Proposition 4. *A Kannan-type aggregate pairwise distance mapping on four points is continuous at its fixed points.*

Proof. Let (X, d) be a metric space with $|X| \geqslant 4$, and let $T : X \to X$ be a Kannan-type aggregate pairwise distance mapping on four points with constant $\lambda \in [0, 3/4)$. Let $x^* \in X$ be a fixed point of T, i.e., $Tx^* = x^*$.

If x^* is isolated, continuity is trivial. Assume that x^* is an accumulation point. It suffices to test continuity using sequences (x_n) with $x_n \to x^*$, all terms mutually distinct, and $x_n \neq x^*$ for all n. Such sequences exist since x^* is an accumulation point.

Applying (4) to the four pairwise distinct points x^*, x_n, x_{n+1}, x_{n+2} and using $Tx^* = x^*$, $d(x^*, Tx^*) = 0$, we obtain

$$\begin{aligned} &d(x^*, Tx_n) + d(x^*, Tx_{n+1}) + d(x^*, Tx_{n+2}) \\ &\quad + d(Tx_n, Tx_{n+1}) + d(Tx_n, Tx_{n+2}) + d(Tx_{n+1}, Tx_{n+2}) \\ &\quad \leqslant \lambda\bigl(d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + d(x_{n+2}, Tx_{n+2})\bigr). \end{aligned} \tag{9}$$

Dropping the nonnegative terms $d(Tx_n, Tx_{n+1})$, $d(Tx_n, Tx_{n+2})$, and $d(Tx_{n+1}, Tx_{n+2})$ from the left-hand side of (9), we get

$$\sum_{i=0}^{2} d(x^*, Tx_{n+i}) \leqslant \lambda \sum_{i=0}^{2} d(x_{n+i}, Tx_{n+i}). \tag{10}$$

Applying the triangle inequality $d(x_{n+i}, Tx_{n+i}) \leqslant d(x_{n+i}, x^*) + d(x^*, Tx_{n+i})$ for $i = 0, 1, 2$ and substituting into (10), we obtain

$$\sum_{i=0}^{2} d(x^*, Tx_{n+i}) \leqslant \lambda \sum_{i=0}^{2} \bigl(d(x_{n+i}, x^*) + d(x^*, Tx_{n+i})\bigr).$$

Rearranging and using $\lambda < 3/4 < 1$ so that $1 - \lambda > 0$,

$$\sum_{i=0}^{2} d(x^*, Tx_{n+i}) \leqslant \frac{\lambda}{1-\lambda} \sum_{i=0}^{2} d(x_{n+i}, x^*). \tag{11}$$

Since the left-hand side of (11) contains $d(x^*, Tx_n)$, we have

$$d(x^*, Tx_n) \leqslant \frac{\lambda}{1-\lambda}\bigl(d(x_n, x^*) + d(x_{n+1}, x^*) + d(x_{n+2}, x^*)\bigr).$$

As $n \to \infty$, $d(x_{n+i}, x^*) \to 0$ for $i = 0, 1, 2$, so $d(x^*, Tx_n) \to 0$, i.e., $Tx_n \to x^* = Tx^*$. Hence T is continuous at x^*. □

4 Fixed point theorems for asymptotically regular mappings and approximating fixed point sequence

The notions of asymptotic regularity and approximating fixed point sequences broaden the applicability of contractive mappings, allowing fixed point theorems to hold for a broader class of mappings.

In this section, we extend Theorem 1 in two directions. In the first approach, we enlarge the class of admissible constants considered in (4) by introducing functional generalizations of Kannan-type aggregate pairwise distance mappings on four points, namely the G-Kannan-type and $\mathcal{B}$-Kannan-type aggregate pairwise distance mappings. Under the assumptions of asymptotic regularity or approximating fixed point sequences, together with continuity or weaker forms such as asymptotic k-continuity, these extensions provide a study of functional versions of Kannan-type aggregate pairwise distance mappings on four points (Theorems 2 and 3). These results encompass classes that permit λ to range over $[0, \infty)$ (Corollaries 2 and 3). In the second approach, we assume only asymptotic regularity or approximating fixed point sequences, which allows the constant λ in (4) to be improved from $[0, 3/4)$ to $[0, 1)$ (Theorem 4).

Let (X, d) be a metric space. A mapping $T : X \to X$ is called asymptotically regular [10] if it satisfies

$$\lim_{n\to\infty} d\big(T^{n+1}(x), T^n(x)\big) = 0$$

for all $x \in X$. A sequence $\{x_n\} \subset X$ is called an *approximate fixed point sequence* of T if $d(x_n, Tx_n) \to 0$ as $n \to \infty$.

Remark 5. Let (X, d) be a metric space, and let $T : X \to X$ be a self-mapping. Consider an initial point $x_0 \in X$ and define the sequence $\{x_n\}$ iteratively by $x_{n+1} = T(x_n)$ for all $n \geqslant 0$. If T is asymptotically regular for all $x \in X$ and the sequence $\{x_n\}$ does not possess a fixed point of T, then all the points x_i, $i \geqslant 0$, are distinct. Indeed, if any two points in the sequence coincide, the sequence $\{x_n\}$ would eventually become periodic, violating the asymptotic regularity condition.

Let (X, d) be a metric space, and let $T : X \to X$ be a self-mapping. The *orbit* of a point $x_0 \in X$ under T is defined by

$$O(T, x_0) = \big\{x_0, T(x_0), T^2(x_0), \dots, T^n(x_0), \dots\big\}.$$

We now recall some weaker notions of continuity for mappings on metric spaces.

Definition 5. Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be

(i) k-continuous [23] for a fixed integer $k \geqslant 1$ if, for every sequence $\{x_n\}$ in X, the convergence $T^{k-1}(x_n) \to t$ implies $T^k(x_n) \to T(t)$ as $n \to \infty$;

(ii) asymptotically k-continuous [24] if, for every sequence $\{x_n\}$ in X satisfying $\lim_{k,n\to\infty} T^k(x_n) = t$, it follows that $\lim_{k,n\to\infty} T(T^k(x_n)) = T(t)$.

Remark 6. The following relationships among various types of continuity are well known (see [7, 24]):

(i) Every asymptotically k-continuous mapping is k-continuous, and hence continuous.
(ii) A continuous mapping (or a k-continuous mapping) need not be asymptotically k-continuous.
(iii) Continuity implies k-continuity, but the converse need not hold. In particular, 1-continuity is equivalent to continuity.

We now define a more general version of the *Kannan-type aggregate pairwise distance mapping* on four points. First, we introduce the class $\mathcal{G}$ of functions $G : \mathbb{R}^{+4} \to \mathbb{R}^+$ that satisfy the following conditions:

(i) $G(0,0,0,0) = 0$; (ii) G is continuous at $(0,0,0,0)$.

Definition 6. Let (X,d) be a metric space with $|X| \geqslant 4$. We say that a mapping $T : X \to X$ is a *G-Kannan-type aggregate pairwise distance mapping on four points* if there exists $G \in \mathcal{G}$ such that the inequality

$$\sum_{1\leqslant i<j\leqslant 4} d(Tx_i, Tx_j) \leqslant G\bigl(d(x_1,Tx_1), d(x_2,Tx_2), d(x_3,Tx_3), d(x_4,Tx_4)\bigr) \qquad (12)$$

holds for all four pairwise distinct points $x_1, x_2, x_3, x_4 \in X$.

Theorem 2. *Let (X,d) be a complete metric space with $|X| \geqslant 4$, and let $T : X \to X$ be an asymptotically k-continuous, asymptotically regular G-Kannan-type aggregate pairwise distance mapping on four points. Then T has a fixed point, and the number of fixed points is at most three.*

Proof. Let $x_0 \in X$ and define the iterative sequence $x_n = T(x_{n-1}) = T^{n-1}x_0$. Suppose that $\{x_n\}$ does not converge to a fixed point of T. We aim to show that $\{x_n\}$ is a Cauchy sequence. It suffices to prove that for all $p > 0$,

$$d(x_n, x_{n+p}) \to 0 \quad \text{as } n \to \infty.$$

For $p = 1$, this follows directly from the definition of asymptotic regularity. For $p = 2$, the triangle inequality gives

$$d(x_n, x_{n+2}) \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}).$$

Applying asymptotic regularity, we obtain

$$d(x_n, x_{n+2}) \to 0 \quad \text{as } n \to \infty.$$

For $p \geqslant 3$, Remark 5 ensures that the points x_n, x_{n+p-2}, x_{n+p-1}, and x_{n+p} are pairwise distinct. Using repeated applications of the triangle inequality, inequality (12), and asymptotic regularity, we obtain

$$\begin{aligned} &d(x_n, x_{n+p}) \\ &\quad \leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+p+1}) + d(x_{n+p+1}, x_{n+p}) \\ &\qquad + d(x_{n+1}, x_{n+p}) + d(x_{n+1}, x_{n+p-1}) + d(x_{n+p+1}, x_{n+p-1}) + d(x_{n+p}, x_{n+p-1}) \end{aligned}$$

$$\begin{aligned}
&\leqslant d(x_n, x_{n+1}) \\
&\quad + G\bigl(d(x_n, Tx_n), d(x_{n+p}, Tx_{n+p}), d(x_{n+p-1}, Tx_{n+p-1}), d(x_{n+p-2}, Tx_{n+p-2})\bigr) \\
&= d(x_n, x_{n+1}) \\
&\quad + G\bigl(d(x_n, x_{n+1}), d(x_{n+p}, x_{n+p+1}), d(x_{n+p-1}, x_{n+p}), d(x_{n+p-2}, x_{n+p-1})\bigr) \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Thus, $\{x_n\}$ is a Cauchy sequence. By the completeness of (X, d), there exists a point $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Moreover, for each integer $k \geqslant 1$, we have $\lim_{n\to\infty} T^k x_n = x^*$, and in particular,

$$\lim_{k,n\to\infty} T^k x_n = x^*.$$

Suppose that T is asymptotically k-continuous. Then, by the definition of asymptotic k-continuity and the above limit, it follows that

$$\lim_{k,n\to\infty} T\bigl(T^k x_n\bigr) = T(x^*).$$

On the other hand, since $\lim_{k,n\to\infty} T^{k+1} x_n = x^*$, we conclude that

$$T(x^*) = x^*,$$

and hence x^* is a fixed point of T. □

There exist mappings that admit an approximate fixed point sequence without satisfying the asymptotic regularity condition (see [27]). The following theorem shows that, for G-Kannan-type mappings, an approximate fixed point sequence is sufficient to guarantee a fixed point.

Theorem 3. *Let (X, d) be a complete metric space with $|X| \geqslant 4$, and let $T : X \to X$ be an asymptotically k-continuous, G-Kannan-type aggregate pairwise distance mapping on four points. Suppose that T has an approximate fixed point sequence. Then T has a fixed point, and the number of fixed points is at most three.*

Proof. Let $x_0 \in X$. Define $x_n = T(x_{n-1}) = T^{n-1} x_0$. Using repeated applications of the triangle inequality, inequality (12), and the approximate fixed point sequence assumption, we obtain

$$\begin{aligned}
&d(x_n, x_{n+p}) \\
&\quad \leqslant d(x_n, Tx_n) + d(Tx_n, Tx_{n+p}) + d(Tx_{n+p}, x_{n+p}) \\
&\quad \leqslant d(x_n, Tx_n) + d(Tx_n, Tx_{n+p}) + d(Tx_{n+p}, Tx_{n+p-1}) + d(Tx_n, Tx_{n+p-1}) \\
&\qquad + d(Tx_n, Tx_{n+p-2}) + d(Tx_{n+p}, Tx_{n+p-2}) + d(Tx_{n+p-1}, Tx_{n+p-2}) \\
&\quad \leqslant d(x_n, x_{n+1}) \\
&\qquad + G\bigl(d(x_n, Tx_n), d(x_{n+p}, Tx_{n+p}), d(x_{n+p-1}, Tx_{n+p-1}), d(x_{n+p-2}, Tx_{n+p-2})\bigr) \\
&\quad \to 0
\end{aligned}$$

as $n \to \infty$. Thus, $\{x_n\}$ is a Cauchy sequence. The rest of the proof follows easily. □

In the next theorem, we show that dropping the requirement of continuity or asymptotic k-continuity allows us to compensate by restricting the constant λ to the interval $[0,1)$ instead of $[0,\infty)$.

Theorem 4. *Let (X,d) be a complete metric space with $|X| \geqslant 4$, and let $T: X \to X$ be an asymptotically regular Kannan-type aggregate pairwise distance mapping on four points with coefficient $\lambda \in [0,1)$. Then T has a fixed point, and the number of fixed points is at most three.*

Proof. Taking

$$G(t_1,t_2,t_3,t_4) = \lambda(t_1+t_2+t_3+t_4), \quad \lambda \in [0,1),$$

inequality (12) reduces to the Kannan-type aggregate pairwise distance condition with coefficient λ. The conclusion now follows from Theorem 2 up to the Cauchy sequence argument, together with (8). □

Let $\mathcal{B}$ denote the set of functions $\beta: [0,\infty) \to [0,\infty)$ satisfying the condition

$$\limsup_{t\to 0} \beta(t) < \infty.$$

Definition 7. Let (X,d) be a metric space with $|X| \geqslant 4$. We say that a mapping $T: X \to X$ is a *$\mathcal{B}$-Kannan-type aggregate pairwise distance mapping on four points* in X if there exist functions $\beta_i \in \mathcal{B}$, $i=1,2,3,4$, such that the inequality

$$\sum_{1\leqslant i<j\leqslant 4} d(Tx_i,Tx_j) \leqslant \sum_{i=1}^{4} \beta_i\bigl(d(x_i,Tx_i)\bigr) d(x_i,Tx_i) \tag{13}$$

holds for all four pairwise distinct points $x_1,x_2,x_3,x_4 \in X$.

Corollary 2. *Let (X,d) be a complete metric space with $|X| \geqslant 4$, and let $T: X \to X$ be an asymptotically k-continuous, asymptotically regular $\mathcal{B}$-Kannan-type aggregate pairwise distance mapping on four points. Then T has a fixed point. Moreover, the number of fixed points is at most three.*

Proof. Define

$$G(x_1,x_2,x_3,x_4) = \beta_1(x_1)x_1 + \beta_2(x_2)x_2 + \beta_3(x_3)x_3 + \beta_4(x_4)x_4.$$

Then we have $G(0,0,0,0)=0$, and the condition $\limsup_{t\to 0}\beta_i(t) < \infty$ for $i=1,2,3,4$ implies that

$$\lim_{x_1,x_2,x_3,x_4\to 0} G(x_1,x_2,x_3,x_4) = 0.$$

Hence, the result follows from Theorem 2. □

By setting $\beta_1(t) = \beta_2(t) = \beta_3(t) = \beta_4(t) = \lambda$ with $\lambda \geqslant 0$ in (13), we obtain a generalized Kannan-type aggregate pairwise distance mapping on four points with the coefficient $\lambda \in [0,\infty)$. Hence, we immediately derive the following result.

Corollary 3. *Let (X, d) be a complete metric space with $|X| \geqslant 4$, and let $T : X \to X$ be an asymptotically k-continuous, asymptotically regular generalized Kannan-type aggregate pairwise distance mapping on four points with the coefficient $\lambda \in [0, \infty)$. Then T has a fixed point. Moreover, the number of fixed points is at most three.*

Remark 7. Theorems 2–4, along with their respective corollaries, remain valid if we replace Kannan-type aggregate pairwise distance on a quadrilateral with Kannan-type perimetric contractions on quadrilaterals.

5 Applications

Following [13], we apply Theorem 1 to study the existence of solutions to fractional differential equations with boundary conditions.

5.1 Application in fractional boundary value problems

Let $X = C([0, 1], \mathbb{R})$ be the space of continuous real-valued functions on $[0, 1]$ with the maximum norm

$$\|u\| = \max_{t \in [0,1]} \bigl|u(t)\bigr|.$$

Definition 8. [28] For a function $u(t)$ defined on $[a, b]$, the Caputo fractional derivative of order $\alpha > 0$ is defined as

$${}^cD^{\alpha}_{a+} u(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - s)^{n-\alpha-1} u^{(n)}(s) \, \mathrm{d}s,$$

where $n - 1 < \alpha < n$, and Γ is the gamma function.

We consider the following fractional boundary value problem:

$$\begin{aligned} &{}^cD^{\alpha}_{0+} u(t) = f\bigl(t, u(t)\bigr), \quad t \in [0, 1],\ 3 < \alpha \leqslant 4, \\ &u(0) = A, \qquad u'(0) = B, \qquad u''(0) = C, \qquad u'''(1) = D, \end{aligned} \tag{14}$$

where $f : [0, 1] \times \mathbb{R} \to \mathbb{R}$ is continuous, and $A, B, C, D \in \mathbb{R}$.

Definition 9. A function $u \in C^4([0, 1], \mathbb{R})$ is a solution of (14) if it satisfies the differential equation ${}^cD^{\alpha}_{0+}(u(t)) = f(t, u(t))$ on $[0, 1]$ together with the boundary conditions $u(0) = A$, $u'(0) = B$, $u''(0) = C$, and $u'''(1) = D$.

Lemma 1. *Let $3 < \alpha \leqslant 4$, and let $g : [0, 1] \to \mathbb{R}$ be continuous. A function u is a solution of the fractional integral equation*

$$\begin{aligned} u(t) = {} & \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha-1} g(s) \, \mathrm{d}s - \frac{t^3}{6\Gamma(\alpha - 3)} \int_0^1 (1 - s)^{\alpha-4} g(s) \, \mathrm{d}s \\ & + A + Bt + \frac{C}{2} t^2 + \frac{D}{6} t^3 \end{aligned}$$

if and only if u *is a solution to the fractional boundary value problem*

$$\begin{aligned} &{}^{c}D_{0+}^{\alpha}\big(u(t)\big) = g(t),\\ &u(0) = A, \qquad u'(0) = B, \qquad u''(0) = C, \qquad u'''(1) = D. \end{aligned}$$

Proof. The proof follows the same structure as that of Lemma 3.4 in [2], suitably adapted to the case $3 < \alpha \leqslant 4$ with four boundary conditions. □

Lemma 2. *Let* $x_1(t), x_2(t), x_3(t), x_4(t), x_5(t), x_6(t) \in C[0,1]$. *Then the following inequality holds*:

$$\begin{aligned} &\max_{t\in[0,1]}\big|x_1(t)\big| + \max_{t\in[0,1]}\big|x_2(t)\big| + \max_{t\in[0,1]}\big|x_3(t)\big|\\ &\quad + \max_{t\in[0,1]}\big|x_4(t)\big| + \max_{t\in[0,1]}\big|x_5(t)\big| + \max_{t\in[0,1]}\big|x_6(t)\big|\\ &\leqslant 6\max_{t\in[0,1]}\big(\big|x_1(t)\big| + \big|x_2(t)\big| + \big|x_3(t)\big| + \big|x_4(t)\big| + \big|x_5(t)\big| + \big|x_6(t)\big|\big). \end{aligned} \tag{15}$$

Proof. The proof follows easily from the fact that each summand on the left-hand side of inequality (15) does not exceed

$$\max_{t\in[0,1]}\big(\big|x_1(t)\big| + \big|x_2(t)\big| + \big|x_3(t)\big| + \big|x_4(t)\big| + \big|x_5(t)\big| + \big|x_6(t)\big|\big).$$ □

Define the operator $T : X \to X$ by

$$\begin{aligned} Tu(t) = {}& \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\big(s,u(s)\big)\,\mathrm{d}s - \frac{t^3}{6\Gamma(\alpha-3)}\int_0^1 (1-s)^{\alpha-4} f\big(s,u(s)\big)\,\mathrm{d}s\\ &+ A + Bt + \frac{C}{2}t^2 + \frac{D}{6}t^3. \end{aligned} \tag{16}$$

By the above Lemma 1, the fixed points of T correspond exactly to solutions of the boundary value problem (14).

Remark 8. The constant M, defined in (17) and used in Theorem 5, is obtained by estimating the integral part of the operator $T : X \to X$ given in (16) in the maximum norm $\|\cdot\|$ of $X = C([0,1],\mathbb{R})$. More precisely, M is obtained by taking the supremum over $t \in [0,1]$ in (16) and estimating separately the two integral terms appearing therein. These sup–norm estimates yield

$$M = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{6\Gamma(\alpha-2)}.$$

Theorem 5. *Let* $3 < \alpha \leqslant 4$,

$$M = \frac{1}{\Gamma(\alpha+1)} + \frac{1}{6\Gamma(\alpha-2)}, \tag{17}$$

and let $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ *be a function continuous in its two variables such that for all pairwise distinct functions* $u_1, u_2, u_3, u_4 \in X$ *and for all* $t \in [0,1]$, *the following* *inequality holds*:

$$\begin{aligned}&\bigl|f\bigl(t,u_1(t)\bigr)-f\bigl(t,u_2(t)\bigr)\bigr|+\bigl|f\bigl(t,u_1(t)\bigr)-f\bigl(t,u_3(t)\bigr)\bigr|\\&\quad+\bigl|f\bigl(t,u_1(t)\bigr)-f\bigl(t,u_4(t)\bigr)\bigr|+\bigl|f\bigl(t,u_2(t)\bigr)-f\bigl(t,u_3(t)\bigr)\bigr|\\&\quad+\bigl|f\bigl(t,u_2(t)\bigr)-f\bigl(t,u_4(t)\bigr)\bigr|+\bigl|f\bigl(t,u_3(t)\bigr)-f\bigl(t,u_4(t)\bigr)\bigr|\\&\leqslant K\bigl[\bigl|u_1(t)-Tu_1(t)\bigr|+\bigl|u_2(t)-Tu_2(t)\bigr|\\&\quad+\bigl|u_3(t)-Tu_3(t)\bigr|+\bigl|u_4(t)-Tu_4(t)\bigr|\bigr],\end{aligned}\tag{18}$$

where $K>0$ is such that $KM<1/8$. If T has no periodic points of prime period 2 *and* 3*, then the boundary value problem* (14) *has at least one solution and at most three solutions.*

Proof. We prove that under assumption (18), T is a Kannan-type aggregate pairwise distance mapping. Let $u_1,u_2,u_3,u_4\in X$ be pairwise distinct. Since f is continuous in both of its variables, we have that $Tu_1,Tu_2,Tu_3,Tu_4\in X$.

Using the definition of T, for each $t\in[0,1]$, we have

$$\begin{aligned}&\bigl|Tu_1(t)-Tu_2(t)\bigr|+\bigl|Tu_1(t)-Tu_3(t)\bigr|+\bigl|Tu_1(t)-Tu_4(t)\bigr|\\&\quad+\bigl|Tu_2(t)-Tu_3(t)\bigr|+\bigl|Tu_2(t)-Tu_4(t)\bigr|+\bigl|Tu_3(t)-Tu_4(t)\bigr|\\&\leqslant\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{1\leqslant i<j\leqslant 4}\bigl|f\bigl(s,u_i(s)\bigr)-f\bigl(s,u_j(s)\bigr)\bigr|\,\mathrm{d}s\\&\quad+\frac{t^3}{6\,\Gamma(\alpha-3)}\int_0^1(1-s)^{\alpha-4}\sum_{1\leqslant i<j\leqslant 4}\bigl|f\bigl(s,u_i(s)\bigr)-f\bigl(s,u_j(s)\bigr)\bigr|\,\mathrm{d}s.\end{aligned}$$

Applying condition (18) at each point $s\in[0,1]$, we obtain

$$\begin{aligned}&\bigl|Tu_1(t)-Tu_2(t)\bigr|+\bigl|Tu_1(t)-Tu_3(t)\bigr|+\bigl|Tu_1(t)-Tu_4(t)\bigr|\\&\quad+\bigl|Tu_2(t)-Tu_3(t)\bigr|+\bigl|Tu_2(t)-Tu_4(t)\bigr|+\bigl|Tu_3(t)-Tu_4(t)\bigr|\\&\leqslant K\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\sum_{i=1}^4\bigl|u_i(s)-Tu_i(s)\bigr|\,\mathrm{d}s\\&\quad+K\frac{t^3}{6\Gamma(\alpha-3)}\int_0^1(1-s)^{\alpha-4}\sum_{i=1}^4\bigl|u_i(s)-Tu_i(s)\bigr|\,\mathrm{d}s.\end{aligned}$$

Since $|u_i(s)-Tu_i(s)|\leqslant\|u_i-Tu_i\|$ for all $s\in[0,1]$ and $i=1,2,3,4$, we have

$$\begin{aligned}&\bigl|Tu_1(t)-Tu_2(t)\bigr|+\bigl|Tu_1(t)-Tu_3(t)\bigr|+\bigl|Tu_1(t)-Tu_4(t)\bigr|\\&\quad+\bigl|Tu_2(t)-Tu_3(t)\bigr|+\bigl|Tu_2(t)-Tu_4(t)\bigr|+\bigl|Tu_3(t)-Tu_4(t)\bigr|\\&\leqslant K\sum_{i=1}^4\bigl\|u_i-Tu_i\bigr\|\left(\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\,\mathrm{d}s+\frac{t^3}{6\Gamma(\alpha-3)}\int_0^1(1-s)^{\alpha-4}\,\mathrm{d}s\right).\end{aligned}$$

Evaluating the integrals

$$\int_0^t (t-s)^{\alpha-1}\,\mathrm{d}s = \frac{t^\alpha}{\alpha}, \qquad \int_0^1 (1-s)^{\alpha-4}\,\mathrm{d}s = \frac{1}{\alpha-3}$$

and noting that $t^\alpha \leqslant 1$, $t^3 \leqslant 1$ for each $t \in [0,1]$, we get

$$\begin{aligned}&\left|Tu_1(t) - Tu_2(t)\right| + \left|Tu_1(t) - Tu_3(t)\right| + \left|Tu_1(t) - Tu_4(t)\right| \\ &\quad + \left|Tu_2(t) - Tu_3(t)\right| + \left|Tu_2(t) - Tu_4(t)\right| + \left|Tu_3(t) - Tu_4(t)\right| \\ &\leqslant K \sum_{i=1}^4 \left\|u_i - Tu_i\right\| \left(\frac{1}{\alpha\Gamma(\alpha)} + \frac{1}{6(\alpha-3)\Gamma(\alpha-3)}\right).\end{aligned}$$

Since the above inequality holds for each $t \in [0,1]$, it also holds if the left-hand side is replaced by its maximum over $t \in [0,1]$. Using the identities $\alpha\Gamma(\alpha) = \Gamma(\alpha+1)$ and $(\alpha-3)\Gamma(\alpha-3) = \Gamma(\alpha-2)$ and the definition of M from (17), we obtain

$$\max_{t\in[0,1]} \left(\sum_{1\leqslant i<j\leqslant 4} \left|Tu_i(t) - Tu_j(t)\right|\right) \leqslant KM \sum_{i=1}^4 \|u_i - Tu_i\|.$$

Now applying Lemma 2 to the six pairwise distance functions, we get

$$\begin{aligned}\sum_{1\leqslant i<j\leqslant 4} \|Tu_i - Tu_j\| &\leqslant 6 \max_{t\in[0,1]} \left(\sum_{1\leqslant i<j\leqslant 4} \left|Tu_i(t) - Tu_j(t)\right|\right) \\ &\leqslant 6KM \sum_{i=1}^4 \|u_i - Tu_i\|.\end{aligned}$$

Thus, T is a Kannan-type aggregate pairwise distance mapping with the parameter $\lambda = 6KM$. Since $6KM < 3/4$ by assumption, we have $\lambda \in [0, 3/4)$. By Theorem 1, since X is complete and T has no periodic points of prime period 2 and 3, T has at least one fixed point and at most three fixed points. Therefore, the boundary value problem (14) has at least one solution and at most three solutions. □

In the next subsection, following [24], we use the sequences from Examples 4 and 5 to derive solutions of a nonlinear Diophantine equation

5.2 Application in the solution of a nonlinear Diophantine equation

The sequences $\{m_i\}$ and $\{n_i\}$, presented in Examples 4 and 5 as illustrations of Theorem 1, generate infinitely many Pythagorean triples with consecutive integer legs. This reveals a noteworthy connection between fixed point theory, functional equations, and number theory; see [24], which marks the first work in this direction.

Theorem 6. *The Fibonacci-type sequences* $\{n_i\}$ *and* $\{m_i\}$ *from the above Examples* 4 *and* 5 *yield infinitely integral solutions of the Diophantine equation*

$$X^2 + (X+1)^2 = Y^2.$$

That is, $m_i^2 + (m_i+1)^2 = n_i^2$ *for all* $i \geqslant 1$.

Proof. A direct verification for the initial terms shows that

$$\begin{aligned} 0^2 + 1^2 &= 1^2 && (m_1 = 0,\ n_1 = 1),\\ 3^2 + 4^2 &= 5^2 && (m_2 = 3,\ n_2 = 5),\\ 20^2 + 21^2 &= 29^2 && (m_3 = 20,\ n_3 = 29),\\ 119^2 + 120^2 &= 169^2 && (m_4 = 119,\ n_4 = 169),\\ 696^2 + 697^2 &= 985^2 && (m_5 = 696,\ n_5 = 985),\\ 4059^2 + 4060^2 &= 5741^2 && (m_6 = 4059,\ n_6 = 5741). \end{aligned}$$

Since both sequences satisfy recurrence relations that preserve the Diophantine identity $m_i^2 + (m_i+1)^2 = n_i^2$, the pattern continues for all $i \geqslant 1$. The general case follows by simultaneous induction for $i \geqslant 3$, using the auxiliary identity

$$n_{i-1}n_{i-2} = 2m_{i-1}m_{i-2} + m_{i-1} + m_{i-2} + 2,$$

which is readily verified from the recurrence relations.

□

6 Conclusion

In this paper, we have introduced a new class of mappings, called Kannan-type aggregate pairwise distance mappings on four points, in the setting of metric spaces. This notion generalizes several well-known concepts, including Kannan mappings, generalized Kannan-type mappings, and Kannan-type perimetric contractions on quadrilaterals. We have investigated the fundamental properties of these mappings and have explored their connections with related concepts. We have further introduced functional generalizations, namely the G-Kannan-type and $\mathcal{B}$-Kannan-type variants, and have shown that, under assumptions of asymptotic regularity or the existence of approximating fixed point sequences, the admissible range of the associated parameter can be extended. Finally, as applications, we have employed the main theorem to establish the existence of solutions for fractional differential equations with boundary conditions and to obtain solutions to a nonlinear Diophantine equation.

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