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A perspective on Banach and Kannan mappings contracting the perimeter of triangles in interpolative metric spaces*

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Abstract. This article delves into several well-known mappings within the framework of interpolative metric spaces. To demonstrate the genuine generalization of metric spaces by interpolative metric spaces, various illustrative examples are provided. The properties of mappings that contract the perimeter of triangles are examined, and a necessary and sufficient condition for the existence of fixed points is established for such mappings, supported by relevant examples. Additionally, the study explores generalized Kannan-type mappings and generalizes a fixed point result of metric spaces in the setting of interpolative metric spaces. Further examples are provided in support of our result. Furthermore, an adequate condition for the uniqueness of the fixed points is proposed.

Keywords: fixed points, interpolative metric spaces, mapping contracting perimeter of triangles, generalized Kannan-type mappings.

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1 Introduction and preliminaries

Metric fixed point theory has been evolving for more than a century. A seminal result that marked the beginning of this productive field was introduced by Banach [3] in 1922. Over the years, researchers have significantly expanded the theory, leading to numerous notable findings. As a result, applications of metric fixed point theory have been established across various quantitative sciences.

The fundamental result has been generalized in two distinct ways. Firstly, by introducing new contraction conditions and verifying whether these imply the existence and uniqueness of the fixed point under a certain environment. Among these, Kannan-type in [9], Ćirić-type in [7], Wardowski-type in [19], enriched-type in [5], weak enriched-type in [14], convex orbital β-Lipschitz-type mappings in [15] are the most notable.

In what follows, we highlight the first research direction, which has served as a key source of inspiration for our own work. In 2023, Petrov [17] introduced a class of mappings characterized by their ability to contract the perimeters of triangles. Let us begin by recalling the definition of such mappings.

Definition 1. (See [17].) Let (X, d) be a metric space with $|X| \geqslant 3$. A mapping $T\colon X \to X$ is defined as mapping contracting perimeters of triangles if there exists a $\gamma \in [0, 1)$ such that

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leqslant \gamma\big[d(x, y) + d(y, z) + d(z, x)\big]$$

for every three pairwise distinct points $x, y, z \in X$.

The author has obtained fixed point results related to these mappings and presented some adequate conditions for the uniqueness of fixed points.

Thereafter, in 2024, Petrov and Bisht [18] introduced the three-point analogue of Kannan-type mappings using the idea of mapping contracting perimeters of triangles and obtained fixed point results. Let us recall the definition of such mappings.

Definition 2. (See [18].) Let (X, d) be a metric space with $|X| \geqslant 3$. A mapping $T\colon X \to X$ is called a generalized Kannan-type mapping if there exists $\beta \in [0, 2/3)$ such that

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leqslant \beta\big[d(x, Tx) + d(y, Ty) + d(z, Tz)\big] \quad (1)$$

for every three pairwise distinct points $x, y, z \in X$.

Building upon these foundational developments, several meaningful extensions of the three-point analogue of several types have been proposed in recent literature. For detailed discussions and further contributions in this direction, we refer the reader to [4, 6, 16].

On the other hand, another fundamental approach to generalizing fixed point theory focuses on weakening the underlying space, while identifying suitable conditions that guarantee the existence of fixed points. Notable examples of such generalized spaces include b-metric spaces in [8], quasimetric spaces in [2], partial metric spaces in [13], metric-like spaces in [1], etc.

In 2023, Karapınar [10] introduced the concept of an interpolative metric by modifying the classical triangle inequality through the inclusion of interpolative-type terms. Interpolative metric spaces constitute a subclass of b-metric spaces, though not all b-metrics are interpolative. A Banach fixed point theorem was established in this setting, and several other significant results have since been obtained; see [11, 12] for further details.

To facilitate a deeper understanding, we now provide the formal definition of the interpolative metric space, followed by illustrative examples and other fundamental notions within this framework. These elements form the essential foundation for the subsequent development of fixed point theory in this setting. We begin with the following definition.

Definition 3. (See [10].) Let X be a nonempty set. A mapping $d : X \times X \to [0, \infty)$ is called an (α, c)-interpolative metric if there exist $\alpha \in (0, 1)$ and $c \geqslant 0$ such that the following conditions holds for all $x, y, z \in X$:

(i) $d(x, y) = 0 \Leftrightarrow x = y$;
(ii) $d(x, y) = d(y, x)$;
(iii) $d(x, y) \leqslant d(x, z) + d(z, y) + c[(d(x, z))^{\alpha}(d(z, y))^{1-\alpha}]$.

Also, (X, d) is called an (α, c)-interpolative metric space.

Remark 1. It is evident that every metric is an (α, c)-interpolative metric for any $\alpha \in (0, 1)$ and $c \geqslant 0$. Moreover, every (α, c)-interpolative metric space (X, d) is a b-metric space studied in [8] with coefficient $s = 1 + c$. Indeed, for any $x, y, z \in X$, we have

$$\begin{aligned} d(x, y) &\leqslant d(x, z) + d(z, y) + c\big[d(x, z)^{\alpha} d(z, y)^{1-\alpha}\big] \\ &\leqslant d(x, z) + d(z, y) + c\big[\big(\max\big\{d(x, z), d(z, y)\big\}\big)^{\alpha}\big(\max\big\{d(x, z), d(z, y)\big\}\big)^{1-\alpha}\big] \\ &\leqslant d(x, z) + d(z, y) + c\max\big\{d(x, z), d(z, y)\big\} \\ &\leqslant (1 + c)\big[d(x, z) + d(z, y)\big]. \end{aligned}$$

Therefore, (X, d) satisfies the b-metric inequality with coefficient $s = 1 + c$.

In the following examples, we show that there exist (α, c)-interpolative metric spaces, which are not metric spaces.

Example 1. Let $X = \mathbb{R}^n$, and let $d' : X \times X \to [0, \infty)$ be defined as

$$d'(x, y) = \max_{1 \leqslant i \leqslant n} |x_i - y_i|^2$$

for all $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n) \in X$.

Now, conditions (i) and (ii) can be easily verified. For each $x = (x_1, x_2, \dots, x_n),\ y = (y_1, y_2, \dots, y_n),\ z = (z_1, z_2, \dots, z_n) \in X$ and for all i with $1 \leqslant i \leqslant n$, we have

$$\begin{aligned} & |x_i - y_i| \leqslant |x_i - z_i| + |z_i - y_i| \\ \Longrightarrow \quad & |x_i - y_i|^2 \leqslant |x_i - z_i|^2 + |z_i - y_i|^2 + 2|x_i - z_i||z_i - y_i| \end{aligned}$$

$$\begin{aligned}
\implies\quad & |x_i - y_i|^2 \leqslant \max_{1\leqslant i\leqslant n} |x_i - z_i|^2 + \max_{1\leqslant i\leqslant n} |z_i - y_i|^2 \\
& \qquad + 2 \max_{1\leqslant i\leqslant n} |x_i - z_i| \max_{1\leqslant i\leqslant n} |z_i - y_i| \\
\implies\quad & \max_{1\leqslant i\leqslant n} |x_i - y_i|^2 \leqslant \max_{1\leqslant i\leqslant n} |x_i - z_i|^2 + \max_{1\leqslant i\leqslant n} |z_i - y_i|^2 \\
& \qquad + 2 \max_{1\leqslant i\leqslant n} |x_i - z_i| \max_{1\leqslant i\leqslant n} |z_i - y_i| \\
\implies\quad & \max_{1\leqslant i\leqslant n} |x_i - y_i|^2 \leqslant \max_{1\leqslant i\leqslant n} |x_i - z_i|^2 + \max_{1\leqslant i\leqslant n} |z_i - y_i|^2 \\
& \qquad + 2\Big[\max_{1\leqslant i\leqslant n} |x_i - z_i|^2\Big]^{1/2} \Big[\max_{1\leqslant i\leqslant n} |z_i - y_i|^2\Big]^{1/2} \\
\implies\quad & d'(x,y) \leqslant d'(x,z) + d'(z,y) + 2\big[\big(d'(x,z)\big)^{1/2}\big(d'(z,y)\big)^{1/2}\big].
\end{aligned}$$

This implies that (iii) holds for $c = 2$ and $\alpha = 1/2$. Thus, (X, d') is a $(1/2, 2)$-interpolative metric space. However, it is an easy exercise to show that (X, d') is not a metric space. Moreover, (X, d') is also a b-metric space with coefficient $s = 1 + c = 3$.

Example 2. Let (X, d) be a metric space. Define $D : X \times X \to [0, \infty)$ as

$$D(x,y) = \frac{d(x,y) + (d(x,y))^2}{1 + d(x,y) + (d(x,y))^2} \quad \text{for all } x, y \in X.$$

It is easy to verify (i) and (ii). Now, for each $x, y, z \in X$, let us denote $d(x, y) = a_{xy}$, $d(x, z) = b_{xz}$, $d(z, y) = c_{zy}$. Then $a_{xy}, b_{xz}, c_{zy} \geqslant 0$ and $a_{xy} \leqslant b_{xz} + c_{zy}$. Moreover, we obtain

$$\begin{aligned}
D(x,y) &= \frac{a_{xy} + a_{xy}^2}{1 + a_{xy} + a_{xy}^2} = 1 - \frac{1}{1 + a_{xy} + a_{xy}^2} \leqslant 1 - \frac{1}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} \\
&= \frac{(b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} \\
&= \frac{b_{xz} + b_{xz}^2}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} + \frac{c_{zy} + c_{zy}^2}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} \\
&\quad + \frac{2b_{xz}c_{zy}}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} \\
&\leqslant \frac{b_{xz} + b_{xz}^2}{1 + b_{xz} + b_{xz}^2} + \frac{c_{zy} + c_{zy}^2}{1 + c_{zy} + c_{zy}^2} + 2\,\frac{b_{xz}c_{zy}}{1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2} \\
&= \frac{b_{xz} + b_{xz}^2}{1 + b_{xz} + b_{xz}^2} + \frac{c_{zy} + c_{zy}^2}{1 + c_{zy} + c_{zy}^2} + 2\bigg(\frac{b_{xz}^2c_{zy}^2}{[1 + (b_{xz}+c_{zy}) + (b_{xz}+c_{zy})^2]^2}\bigg)^{1/2} \\
&\leqslant \frac{b_{xz} + b_{xz}^2}{1 + b_{xz} + b_{xz}^2} + \frac{c_{zy} + c_{zy}^2}{1 + c_{zy} + c_{zy}^2} + 2\bigg(\frac{b_{xz}^2c_{zy}^2}{(1 + b_{xz} + b_{xz}^2)(1 + c_{zy} + c_{zy}^2)}\bigg)^{1/2} \\
&\leqslant \frac{b_{xz} + b_{xz}^2}{1 + b_{xz} + b_{xz}^2} + \frac{c_{zy} + c_{zy}^2}{1 + c_{zy} + c_{zy}^2} + 2\bigg(\frac{(b_{xz} + b_{xz}^2)(c_{zy} + c_{zy}^2)}{(1 + b_{xz} + b_{xz}^2)(1 + c_{zy} + c_{zy}^2)}\bigg)^{1/2}
\end{aligned}$$

$$= \frac{b_{xz} + b_{xz}^2}{1 + b_{xz} + b_{xz}^2} + \frac{c_{zy} + c_{zy}^2}{1 + c_{zy} + c_{zy}^2} + 2\left(\frac{b_{xz}+b_{xz}^2}{1+b_{xz}+b_{xz}^2}\right)^{1/2}\left(\frac{c_{zy}+c_{zy}^2}{1+c_{zy}+c_{zy}^2}\right)^{1/2}$$

$$= \frac{d(x,z) + (d(x,z))^2}{1 + d(x,z) + (d(x,z))^2} + \frac{d(z,y) + (d(z,y))^2}{1 + d(z,y) + (d(z,y))^2}$$

$$+ 2\left(\frac{d(x,z) + (d(x,z))^2}{1 + d(x,z) + (d(x,z))^2}\right)^{1/2}\left(\frac{d(z,y) + (d(z,y))^2}{1 + d(z,y) + (d(z,y))^2}\right)^{1/2}$$

$$= D(x,z) + D(z,y) + 2\big(D(x,z)\big)^{1/2}\big(D(z,y)\big)^{1/2}.$$

This implies that (iii) is satisfied for $c = 2$ and $\alpha = 1/2$. Thus, (X, D) is a $(1/2, 2)$-interpolative metric space. Moreover, (X, D) is also a b-metric space with coefficient $s = 1 + c = 3$.

Example 3. Let $X = C[a,b]$ be the collection of all continuous functions defined on $[a,b] \subseteq \mathbb{R}$, and let $\delta : X \times X \to [0,\infty)$ be defined as

$$\delta(f,g) = \sup_{x\in[a,b]} \big|f(x) - g(x)\big|^2 \quad \text{for all } f, g \in C[a,b].$$

It is easy to see that (i) and (ii) hold.

Now, for each $f, g, h \in C[a,b]$ and for each $x \in [a,b]$, we have

$$\big|f(x) - g(x)\big| \leqslant \big|f(x) - h(x)\big| + \big|h(x) - g(x)\big|$$

$$\Longrightarrow \quad \big|f(x) - g(x)\big|^2 \leqslant \big|f(x) - h(x)\big|^2 + \big|h(x) - g(x)\big|^2 + 2\big|f(x) - h(x)\big|\big|h(x) - g(x)\big|$$

$$\Longrightarrow \quad \big|f(x) - g(x)\big|^2 \leqslant \sup_{x\in[a,b]} \big|f(x) - h(x)\big|^2 + \sup_{x\in[a,b]} \big|h(x) - g(x)\big|^2 + 2 \sup_{x\in[a,b]} \big|f(x) - h(x)\big| \sup_{x\in[a,b]} \big|h(x) - g(x)\big|$$

$$\Longrightarrow \quad \sup_{x\in[a,b]} \big|f(x) - g(x)\big|^2 \leqslant \sup_{x\in[a,b]} \big|f(x) - h(x)\big|^2 + \sup_{x\in[a,b]} \big|h(x) - g(x)\big|^2 + 2 \sup_{x\in[a,b]} \big|f(x) - h(x)\big| \sup_{x\in[a,b]} \big|h(x) - g(x)\big|$$

$$\Longrightarrow \quad \sup_{x\in[a,b]} \big|f(x) - g(x)\big|^2 \leqslant \sup_{x\in[a,b]} \big|f(x) - h(x)\big|^2 + \sup_{x\in[a,b]} \big|h(x) - g(x)\big|^2 + 2\Big[\sup_{x\in[a,b]} \big|f(x) - h(x)\big|^2\Big]^{1/2} \times \Big[\sup_{x\in[a,b]} \big|h(x) - g(x)\big|^2\Big]^{1/2}$$

$$\Longrightarrow \quad \delta(f,g) \leqslant \delta(f,h) + \delta(h,g) + 2\big[\big(\delta(f,h)\big)^{1/2}\big(\delta(h,g)\big)^{1/2}\big].$$

This implies that (iii) holds for $c = 2$ and $\alpha = 1/2$. Thus, (X, δ) is an $(1/2, 2)$-interpolative metric space. However, it is an easy exercise to show that (X, δ) is not a metric space. Moreover, (X, δ) is also a b-metric space with coefficient $s = 1 + c = 3$.

Definition 4. (See [10].) Let (X, d) be an (α, c)-interpolative metric space, and let $\{x_n\}$ be a sequence in X. We say that $\{x_n\}$ converges to x in X if $d(x_n, x) \to 0$ as $n \to \infty$.

Remark 2. From the above definition and (iii) we obtain that the limit of a convergent sequence in an (α, c)-interpolative metric space is unique. Moreover, it is readily observed that every subsequence of a convergent sequence in an (α, c)-interpolative metric space has the same limit as the sequence itself.

Definition 5. (See [10].) Let (X, d) be an (α, c)-interpolative metric space, and let $\{x_n\}$ be a sequence in X. If $\lim_{n\to\infty} \sup\{d(x_n, x_m)\colon m > n\} = 0$, we say that $\{x_n\}$ is a Cauchy sequence in X.

Definition 6. (See [10].) Let (X, d) be an (α, c)-interpolative metric space. We say that (X, d) is a complete (α, c)-interpolative metric space if every Cauchy sequence converges in X.

It is worth noting that mappings contracting the perimeters of triangles, as well as generalized Kannan-type mappings, have been studied in metric spaces. However, these classes remain unexplored in interpolative metric and b-metric spaces. Motivated by this gap, we investigate these mappings in interpolative metric spaces. As noted in Remark 1, the results extend naturally to b-metric structures; however, our examples are constructed in nonmetric spaces. Furthermore, we establish sufficient conditions for the existence and uniqueness of fixed points and compare our findings with existing results to identify possible connections.

2 Mappings contracting perimeters of triangles

Firstly, we investigate the continuity of mappings contracting the perimeters of triangles within interpolative metric spaces, which share the same definition as in classical metric spaces, as given below.

Definition 7. Let (X, d) be an (α, c)-interpolative metric space with $|X| \geqslant 3$. A mapping $T : X \to X$ is defined as mapping contracting perimeters of triangles if there is a $\gamma \in [0, 1)$ such that

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leqslant \gamma\bigl[d(x, y) + d(y, z) + d(z, x)\bigr] \tag{2}$$

for every three pairwise distinct points $x, y, z \in X$.

Remark 3. If inequality (2) is required to hold for all triples (allowing coincidences), then setting $z = x$ implies $2d(Tx, Ty) \leqslant 2\gamma d(x, y)$, and thus T is a Banach contraction with a constant γ. In this case, the three-point framework collapses to the classical two-point formulation and alters the scope.

Moreover, certain notions in interpolative metric spaces, such as isolated points and the continuity of mappings between two interpolative metric spaces, are defined analogously to those in classical metric spaces, as follows:

Definition 8. Let A be a nonempty subset of an interpolative metric space (X, d). A point $x_0 \in A$ is said to be an isolated point of A if there exists $\varepsilon > 0$ such that

$$B(x_0, \varepsilon) \cap A = \{x_0\},$$

where $B(x_0, \varepsilon) := \{y \in X\colon\ d(x_0, y) < \varepsilon\}$ denotes the open ball centered at x_0 with radius ε.

From the above definition it follows that, if x_0 is not an isolated point, then for each $\varepsilon > 0$, there is a point $y \in A$ with $y \neq x_0$ and $d(x_0, y) < \varepsilon$.

Definition 9. A mapping T from an interpolative metric space (X_1, d_1) to another interpolative metric space (X_2, d_2) is said to be continuous at a point $x_0 \in X_1$ if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $x \in X$ with $d_1(x, x_0) < \delta$, we have $d_2(Tx, Tx_0) < \varepsilon$. In addition, a mapping T is said to be continuous if it is continuous at any point in X.

In view of the preceding definition and by employing an argument analogous to that used in classical metric spaces, the following result holds in interpolative metric spaces, mirroring its counterpart in the standard setting:

- If T is continuous at a point $x \in X$ and a sequence $\{x_n\}$ in X converges to x, then the sequence $\{Tx_n\}$ converges to Tx.

Note that the mapping contracting perimeters of triangles is continuous in the setting of interpolative metric spaces under certain assumptions, as established in the following proposition.

Proposition 1. *Let (X, d) be an (α, c)-interpolative metric space with $|X| \geqslant 3$. Assume that for every nonisolated $x_0 \in X$ and every $\delta > 0$, the punctured ball $B(x_0, \delta) \setminus \{x_0\}$ contains at least two distinct points. If $T : X \to X$ is a mapping contracting perimeters of triangles satisfying* (2) *with the contraction constant $\gamma \in [0, 1)$, then T is continuous.*

Proof. Let $x_0 \in X$ and $\varepsilon > 0$ be arbitrary. If x_0 is an isolated point of X, then it follows immediately that T is continuous at x_0.

Suppose that x_0 is not an isolated point of X. Choose $\delta > 0$ with $\gamma\delta < \varepsilon/(4 + c)$. For each $x \in X$ with $d(x, x_0) < \delta$, if $x = x_0$, clearly, $d(Tx, Tx_0) = 0 < \varepsilon$. Otherwise, $x \in B(x_0, \delta) \setminus \{x_0\}$. From the assumption there are $y_1, y_2 \in B(x_0, \delta) \setminus \{x_0\}$ with $y_1 \neq y_2$. This implies that for any $x \in B(x_0, \delta) \setminus \{x_0\}$, at least one of y_1 and y_2 differs from x; denote it by y. Then (x, x_0, y) are pairwise distinct. So the perimeter inequality (2) applies to

$$\begin{aligned} d(Tx, Tx_0) &\leqslant d(Tx, Tx_0) + d(Tx_0, Ty) + d(Ty, Tx) \\ &\leqslant \gamma\bigl[d(x, x_0) + d(x_0, y) + d(y, x)\bigr] \\ &\leqslant \gamma\bigl[2d(x, x_0) + 2d(x_0, y) + c\bigl[\bigl(d(x, x_0)\bigr)^{\alpha}\bigl(d(x_0, y)\bigr)^{1-\alpha}\bigr]\bigr] \end{aligned}$$

$$\begin{aligned}&\leqslant \gamma\big[2d(x,x_0)+2d(x_0,y)+c\max\big\{\big(d(x,x_0)\big),\big(d(x_0,y)\big)\big\}\big]\\&<\gamma(2\delta+2\delta+c\delta)=\gamma\delta(4+c)\\&<\varepsilon.\end{aligned}$$

Hence, T is continuous at x_0. This completes the proof. □

We are now in a position to establish a result that characterizes the necessary and sufficient condition for the existence of fixed points, which is closely related to the concept of a periodic point of prime period $n \in \mathbb{N}$ of a self-mapping T on an interpolative metric space X, that is, a point $x \in X$ satisfies $T^n x = x$, and the smallest such positive integer n is called the prime period of x.

Theorem 1. *Let (X,d) be a complete (α,c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a mapping contracting perimeters of triangles satisfying* (2) *with the contraction constant $\gamma \in [0,1)$. Then T possesses a fixed point in X if and only if T does not attain periodic points of prime period* 2*. Moreover, the number of fixed points of T is at most two.*

Proof. In the first showing, assume that T does not attain periodic points of prime period 2. Let $a_0 \in X$ be chosen arbitrarily. Define $a_n = Ta_{n-1}$ for all $n \in \mathbb{N}$. If a_n is a fixed point of T for some $n \in \mathbb{N}$, then we have nothing to show. Let us assume that a_n is not a fixed point of T for all $n \in \mathbb{N}$. It follows that $a_0 \neq a_1$, $a_1 \neq a_2$, and so on. Also, since T does not attain periodic points of prime period 2, it follows that $a_0 \neq a_2$, $a_1 \neq a_3$, and so on. Therefore, any three consecutive elements of $\{a_n\}$ are distinct. Define

$$\lambda_n = d(a_n,a_{n+1}) + d(a_{n+1},a_{n+2}) + d(a_{n+2},a_n)$$

for all $n \in \mathbb{N}\cup\{0\}$. Now, for each $n \in \mathbb{N}\cup\{0\}$, we have

$$\lambda_{n+1} \leqslant \gamma\lambda_n \leqslant \gamma^{n+1}\lambda_0.$$

Thus,

$$\begin{aligned}\lim_{n\to\infty}\lambda_n = 0 \quad&\Longrightarrow\quad \lim_{n\to\infty}\big\{d(a_n,a_{n+1})+d(a_{n+1},a_{n+2})+d(a_{n+2},a_n)\big\}=0\\&\Longrightarrow\quad \lim_{n\to\infty}d(a_n,a_{n+1})=0=\lim_{n\to\infty}d(a_n,a_{n+2}).\end{aligned}\tag{3}$$

Assume that

$$\lim_{n\to\infty}d(a_n,a_{n+r})=0 \quad \text{for some } r\in\mathbb{N}.$$

For each $n \in \mathbb{N}\cup\{0\}$, we have

$$\begin{aligned}d(a_n,a_{n+r+1}) \leqslant{}& d(a_n,a_{n+r})+d(a_{n+r},a_{n+r+1})\\&+c\big[\big(d(a_n,a_{n+r})\big)^{\alpha}\big(d(a_{n+r},a_{n+r+1})\big)^{1-\alpha}\big].\end{aligned}$$

Therefore,

$$\lim_{n\to\infty}d(a_n,a_{n+r+1})=0.$$

Inductively, we have

$$\lim_{n\to\infty}d(a_n,a_{n+r})=0 \quad \text{for arbitrary but fixed } r\in\mathbb{N}.\tag{4}$$

Let $m, n \in \mathbb{N}$ be such that $m > n$. Take $r = m - n - 1 \geqslant 0$. If $r = 0$, then $m = n + 1$. In this case, we already have

$$\lim_{n\to\infty} d(a_n, a_m) = \lim_{n\to\infty} d(a_n, a_{n+1}) = 0.$$

Now, let $r \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. From (3) and (4) we can say that there exist $k_1, k_2 \in \mathbb{N}$ such that

$$d(a_n, a_{n+1}) < \frac{\varepsilon}{4(c+1)} \quad \text{for all } n > k_1$$

and

$$d(a_n, a_{n+r}) < \frac{\varepsilon}{4(c+1)} \quad \text{for all } n > k_2.$$

Let $k = \max\{k_1, k_2\}$. Now, for each $m \in \mathbb{N}$ with $m > n > k$, we obtain

$$\begin{aligned}
d(a_n, a_m) &\leqslant d(a_n, a_{n+r}) + d(a_{n+r}, a_m) + c\big[\big(d(a_n, a_{n+r})\big)^{\alpha}\big(d(a_{n+r}, a_m)\big)^{1-\alpha}\big] \\
&\leqslant d(a_n, a_{n+r}) + d(a_{n+r}, a_m) + c\max\big\{d(a_n, a_{n+r}), d(a_{n+r}, a_m)\big\} \\
&\leqslant d(a_n, a_{n+r}) + d(a_{n+r}, a_m) + c\big[d(a_n, a_{n+r}) + d(a_{n+r}, a_m)\big] \\
&\leqslant (c+1)\big[d(a_n, a_{n+r}) + d(a_{n+r}, a_m)\big] \\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.
\end{aligned}$$

This implies that

$$\sup\big\{d(a_n, a_m)\colon\ m > n\big\} \leqslant \frac{\varepsilon}{2} < \varepsilon \quad \Longrightarrow \quad \lim_{n\to\infty} \sup\big\{d(a_n, a_m)\colon\ m > n\big\} = 0.$$

Hence, $\{a_n\}$ is a Cauchy sequence in X. Now by completeness of X, there exists $a^* \in X$ such that $a_n \to a^*$ as $n \to \infty$.

Let us prove that $Ta^* = a^*$. For this claim, we will consider the set $A := \{n \in \mathbb{N}\colon a_n = a^*\}$ in the following two cases.

Case 1. If A is an infinite set, then there exists a sequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that

$$a_{n_k+1} = Ta_{n_k} = Ta^* \quad \text{for all } k \in \mathbb{N}.$$

Consequently, $\lim *_{k\to\infty} a_{n_k+1} = Ta^*$. Since $\{a_{n_k+1}\}_{k\in\mathbb{N}}$ is a subsequence of $\{a_n\}$, Remark 2 implies that $Ta^* = a^*$. Therefore, a^* is a fixed point of T.

Case 2. If A is a finite set, then there exists $N \in \mathbb{N}$ such that $a_n \neq a^*$ for all $n \geqslant N$. This implies that for each $n \geqslant N$, by Remark 1 and (2), we have

$$\begin{aligned}
&d(a^*, Ta^*) \\
&\quad \leqslant (1+c)\big[d(a^*, a_{n+1}) + d(a_{n+1}, Ta^*)\big] = (1+c)\big[d(a^*, a_{n+1}) + d(Ta_n, Ta^*)\big] \\
&\quad \leqslant (1+c)\big[d(a^*, a_{n+1}) + d(Ta_n, Ta^*) + d(Ta^*, Ta_{n+1}) + d(Ta_{n+1}, Ta_n)\big] \\
&\quad \leqslant (1+c)\big[d(a^*, a_{n+1}) + \gamma\big[d(a_n, a^*) + d(a^*, a_{n+1}) + d(a_{n+1}, a_n)\big]\big].
\end{aligned}$$

By taking limit as $n \to \infty$ in the above inequality, we obtain $d(a^*, Ta^*) = 0$. This yields that a^* is a fixed point of T.

From both cases we can conclude that T has a fixed point.

On the other hand, suppose that T has a fixed point, say $a \in X$, and assume further that T has a periodic point b of prime period 2. Define $c := Tb$. If $a = b$ or $b = c$, then $Tb = b$, which contradicts the assumption that b is a periodic point of prime period 2. If $a = c$, then $Tb = c = a = Ta = b$, which again contradicts the fact that b is a periodic point of prime period 2. Therefore, we may assume that a, b, and c are pairwise distinct. In this case, applying inequality (2) yields $\gamma \geqslant 1$, contradicting the assumption that $\gamma \in [0, 1)$. Hence, T cannot have a periodic point of prime period 2.

Finally, let p, q, and r be three distinct fixed points of T. Then by applying inequality (2) to these points, it follows that $\gamma \geqslant 1$, which contradicts the assumption that $\gamma \in [0, 1)$. Therefore, T can have at most two fixed points. □

In what follows, we provide the examples to illustrate and support Theorem 1.

Example 4. Let $X = [1/2, 1] \cup \{0, -1/2\}$, and let $\rho : X \times X \to [0, \infty)$ be defined by

$$\rho(x, y) = |x - y|^4 \quad \text{for all } x, y \in X.$$

Then (X, ρ) is a complete $(1/4, 14)$-interpolative metric space. Define a mapping $T\colon X \to X$ by

$$Tx = \begin{cases} x/2 + 1/2 & \text{if } x \in [1/2, 1], \\ 1/2 & \text{if } x = 0, \\ 0 & \text{if } x = -1/2. \end{cases}$$

Then T is a mapping contracting perimeters of triangles with $\gamma \in [3/4, 1)$, and T does not consist of any periodic points of prime period 2. Thus, Theorem 1 guarantees that T has a fixed point. Clearly $\mathrm{Fix}(T) = \{1\}$.

Next, we showcase an example of mapping contracting perimeters of triangles with exactly two fixed points.

Example 5. Let $X = \{1, 2, 3\}$ be a metric space endowed with the Euclidean metric d. Now, define $D : X \times X \to [0, \infty)$ by

$$D(x, y) = \frac{d(x, y) + (d(x, y))^2}{1 + d(x, y) + (d(x, y))^2} \quad \text{for all } x, y \in X.$$

Then (X, D) is a complete $(1/2, 2)$-interpolative metric space (see Example 2). Define a mapping $T : X \to X$ by $T(1) = 1$, $T(2) = 2$, and $T(3) = 2$. Then T is a mapping contracting perimeters of triangles with $\gamma \in [14/23, 1)$, and T does not contain periodic points of prime period 2. Thus, Theorem 1 ensures the existence of a fixed point of T, where $\mathrm{Fix}(T) = \{1, 2\}$.

The following examples show that none of the conditions of Theorem 1 can be relaxed for the existence of fixed points. Example 6 shows that the absence of periodic points of prime period 2 is necessary, whereas Example 7 demonstrates the necessary of the contraction condition.

Example 6. Let $X = \{0, 1, 2\}$, and let $\delta : X \times X \to [0, \infty)$ be defined by

$$\delta(x, y) = |x - y|^3 \quad \text{for all } x, y \in X.$$

Then (X, δ) is a complete $(1/3, 6)$-interpolative metric space. Define a mapping $T\colon X \to X$ by $T(0) = 1$, $T(1) = 2$, and $T(2) = 1$. Then T is a mapping contracting perimeters of triangles with $\gamma \in [1/5, 1)$. Note that 1 and 2 are periodic points of prime period 2 of T, where $\mathrm{Fix}(T) = \emptyset$.

Example 7. Let $X = \{0, 1, 2, 3\}$, and let $d^* : X \times X \to [0, \infty)$ be defined by

$$d^*(x, y) = |x - y|^2 \quad \text{for all } x, y \in X.$$

Then (X, d^*) is a complete $(1/2, 2)$-interpolative metric space. Define a mapping $T\colon X \to X$ by $T(0) = 3$, $T(1) = 2$, $T(2) = 0$, and $T(3) = 1$. Now,

$$\begin{aligned} &d^*\bigl(T(1), T(2)\bigr) + d^*\bigl(T(2), T(3)\bigr) + d^*\bigl(T(3), T(1)\bigr) \\ &\quad = d^*(2, 0) + d^*(0, 1) + d^*(1, 2) = d^*(1, 2) + d^*(2, 3) + d^*(3, 1). \end{aligned}$$

Thus, T is not a mapping contracting perimeters of triangles. Although T does not consist of any periodic points of prime period 2, it has no fixed point.

As illustrated in Example 5, the existence of more than one fixed point is possible. To guarantee the uniqueness of a fixed point, certain additional conditions must be imposed. These conditions are formulated in the next result.

Proposition 2. *Let (X, d) be a complete (α, c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a mapping contracting perimeters of triangles with no periodic point of prime period 2. For $x_0 \in X$, if the infinite iterative sequence x_0, $x_1 = Tx_0$, $x_2 = Tx_1, \dots,$ converges to a point $\xi \in X$ with $\xi \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$, then ξ is the unique fixed point of T.*

Proof. It follows from Theorem 1 that ξ is a fixed point of T. To prove the uniqueness of ξ, let η be another fixed point of T. Then $\eta \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$; otherwise, we have $\xi = \eta$. Therefore, ξ, η, and x_n are all distinct for all $n \in \mathbb{N} \cup \{0\}$. Now, we have

$$\begin{aligned} K_n &= \frac{d(T\xi, T\eta) + d(T\eta, Tx_n) + d(Tx_n, T\xi)}{d(\xi, \eta) + d(\eta, x_n) + d(x_n, \xi)} \\ &= \frac{d(\xi, \eta) + d(\eta, x_{n+1}) + d(x_{n+1}, \xi)}{d(\xi, \eta) + d(\eta, x_n) + d(x_n, \xi)} \\ &\leqslant \alpha \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Taking the limit as $n \to \infty$, we obtain $K_n \to 1$, which yields that $\alpha \geqslant 1$. This leads to a contradiction that $\alpha \in [0, 1)$. Hence, the result follows. □

Now, we provide an alternative proof of the Banach contraction principle in an interpolative metric space using Theorem 1.

Corollary 1. *Let (X, d) be a complete (α, c)-interpolative metric space, and let $T : X \to X$ be a Banach contraction mapping with contraction constant $k \in [0, 1)$. Then T has a unique fixed point in X.*

Proof. If $|X| = 1$, the assertion is trivially satisfied. Now, suppose that $|X| = 2$ with $X = x, y$. Then there are four possible cases for the mapping T as follows:

1. $Tx = x$ and $Ty = y$,
2. $Tx = y$ and $Ty = x$,
3. $Tx = x$ and $Ty = x$,
4. $Tx = y$ and $Ty = y$.

The first two cases contradict the Banach contractive condition and hence cannot occur. In the remaining two cases, the mapping T has a unique fixed point.

Now, suppose that $|X| \geqslant 3$. If there exists $x \in X$ with $x \neq Tx$ such that $T^2x = x$, then $d(x, Tx) = d(T^2x, Tx) = d(Tx, T^2x) \leqslant kd(x, Tx)$, which contradicts that $k \in [0, 1)$. So, T has no periodic points of prime period 2.

Now, for all distinct points $x, y, z \in X$, we have

$$\begin{aligned} d(Tx, Ty) &\leqslant k\bigl(d(x, y)\bigr), \\ d(Ty, Tz) &\leqslant k\bigl(d(y, z)\bigr), \\ d(Tz, Tx) &\leqslant k\bigl(d(z, x)\bigr). \end{aligned}$$

Therefore, adding these inequalities, we get

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leqslant k\bigl[d(x, y) + d(y, z) + d(z, x)\bigr].$$

Thus, T is a mapping contracting perimeters of triangles on X. Then by Theorem 1 it follows that T admits fixed points. The contraction condition guarantees the uniqueness of the fixed point. □

3 Generalized Kannan-type mappings

Firstly, we introduce the concept of a generalized Kannan-type mapping within interpolative metric spaces as given below.

Definition 10. Let (X, d) be an (α, c)-interpolative metric space with $|X| \geqslant 3$. A mapping $T : X \to X$ is called a generalized Kannan-type mapping if there exists $\beta \in [0, (c + 2)/(3(c + 1)))$ such that

$$d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx) \leqslant \beta\bigl[d(x, Tx) + d(y, Ty) + d(z, Tz)\bigr] \tag{5}$$

for every three pairwise distinct points $x, y, z \in X$.

Next, we would like to check the continuity of generalized Kannan-type mapping in interpolative metric spaces.

Proposition 3. *Let (X, d) be an (α, c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a generalized Kannan-type mapping with contraction constant β. If a^* is a fixed point of T, then for any sequence $\{x_n\}$ with $x_n \to a^*$ as $n \to \infty$, we have $Tx_n \to Ta^*$ as $n \to \infty$, provided that*

$$\beta < \frac{1}{1 + c(1 - \alpha)}.$$

Proof. We will prove this proposition by dividing into three cases.

Case 1. Assume that $x_n = a^*$ for some $n = N \in \mathbb{N}$. Then, clearly, $Tx_n \to Ta^*$ as $n \to \infty$.

Case 2. Assume that $x_n \neq a^*$ and $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. From (5) and by applying Young's inequality, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d(Ta^**, Tx_n) + d(Tx_{n+1}, Ta^*) \\
&\quad \leqslant d(Ta^*, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Ta^*) \\
&\quad \leqslant \beta\bigl[d(a^*, Ta^*) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\bigr] \\
&\quad = \beta\bigl[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\bigr] \\
&\quad \leqslant \beta\bigl\{\bigl[d(x_n, a^*) + d(a^*, Tx_n) + c\bigl[d(x_n, a^*)\bigr]^\alpha \bigl[d(a^*, Tx_n)\bigr]^{1-\alpha}\bigr] \\
&\qquad + \bigl[d(x_{n+1}, a^*) + d(a^*, Tx_{n+1}) + c\bigl[d(x_{n+1}, a^*)\bigr]^\alpha \bigl[d(a^*, Tx_{n+1})\bigr]^{1-\alpha}\bigr]\bigr\} \\
&\quad \leqslant \beta\bigl\{\bigl[d(x_n, a^*) + d(a^*, Tx_n) + c\bigl[\alpha d(x_n, a^*) + (1 - \alpha) d(a^*, Tx_n)\bigr]\bigr] \\
&\qquad + \bigl[d(x_{n+1}, a^*) + d(a^*, Tx_{n+1}) + c\bigl[\alpha d(x_{n+1}, a^*) + (1 - \alpha) d(a^*, Tx_{n+1})\bigr]\bigr]\bigr\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\bigl[1 - \beta\bigl(1 + c(1 - \alpha)\bigr)\bigr]\bigl[d(Ta^*, Tx_n) + d(Tx_{n+1}, Ta^*)\bigr] \\
&\quad \leqslant \beta(1 + c\alpha)\bigl[d(x_n, a^*) + d(x_{n+1}, a^*)\bigr].
\end{aligned}$$

According to the assumption, $d(x_n, a^*) \to 0$ as $n \to \infty$, and $d(x_{n+1}, a^*) \to 0$ as $n \to \infty$. Thus,

$$d(Ta^*, Tx_n) + d(Tx_{n+1}, Ta^*) \to 0 \quad \text{as } n \to \infty.$$

This implies that $d(Ta^*, Tx_n) \to 0$ as $n \to \infty$, i.e., $Tx_n \to Ta^*$ as $n \to \infty$.

Case 3. Assume that $x_n \neq a^*$ for all $n \in \mathbb{N}$, but $x_n = x_{n+1}$ may occur for some $n \in \mathbb{N}$. Construct a subsequence $\{x_{n_p}\}$ considering consecutive distinct terms. Then $x_n \to a^*$ implies $x_{n_p} \to a^*$ as every subsequence has the same limit. Thus, by the previous case $Tx_{n_p} \to Ta^*$ as $p \to \infty$. Due to the construction of the sequence $\{x_{n_p}\}$, the inclusion of repeated terms does not affect the convergence of the sequence $\{Tx_{n_p}\}$. Thus, $Tx_n \to Ta^*$ as $n \to \infty$. Hence, the result follows. □

Proposition 4. *Let (X, d) be an (α, c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a generalized Kannan-type mapping. If a^* is a fixed point of T, then for any sequence $\{x_n\}$ with $x_n \to a^*$ as $n \to \infty$ and $\sup_{n \in \mathbb{N}} d(Tx_n, a^*) < \infty$, we have $Tx_n \to Ta^*$ as $n \to \infty$.*

Proof. We will prove this proposition by dividing into three cases.

Case 1. Assume that $x_n = a^*$ for some $n = N \in \mathbb{N}$. Then, clearly, $Tx_n \to Ta^*$ as $n \to \infty$.

Case 2. Assume that $x_n \neq x_{n+1}$ and $x_n \neq a^*$ for all $n \in \mathbb{N}$. We have to show that $Tx_n \to Ta^*$ as $n \to \infty$. From (5), for each $n \in \mathbb{N}$, we have

$$\begin{aligned}
&d(Ta^*, Tx_n) + d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Ta^*) \\
&\quad \leqslant \beta\bigl[d(a^*, Ta^*) + d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\bigr] \\
&\quad = \beta\bigl[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})\bigr] \\
&\quad \leqslant \beta\bigl\{d(x_n, a^*) + d(a^*, Tx_n) + c\bigl[(d(x_n, a^*))^\alpha\bigl(d(a^*, Tx_n)\bigr)^{1-\alpha}\bigr] \\
&\qquad + d(x_{n+1}, a^*) + d(a^*, Tx_{n+1}) + c\bigl[\bigl(d(x_{n+1}, a^*)\bigr)^\alpha\bigl(d(a^*, Tx_{n+1})\bigr)^{1-\alpha}\bigr]\bigr\}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&(1-\beta)\bigl[d(Ta^*, Tx_n) + d(Tx_{n+1}, Ta^*)\bigr] + d(Tx_n, Tx_{n+1}) \\
&\quad \leqslant \beta\bigl\{d(x_n, a^*) + c\bigl[\bigl(d(x_n, a^*)\bigr)^\alpha\bigl(d(a^*, Tx_n)\bigr)^{1-\alpha}\bigr] \\
&\qquad + d(x_{n+1}, a^*) + c\bigl[\bigl(d(x_{n+1}, a^*)\bigr)^\alpha\bigl(d(a^*, Tx_{n+1})\bigr)^{1-\alpha}\bigr]\bigr\}
\end{aligned}$$

for all $n \in \mathbb{N}$. According to the assumption, $d(x_n, a^*) \to 0$ and $d(x_{n+1}, a^*) \to 0$ as $n \to \infty$, and $\sup_{n\in\mathbb{N}} d(Tx_n, a^*) < \infty$. Thus, $d(Tx_n, Ta^*) \to 0$ as $n \to \infty$, i.e., $Tx_n \to Ta^*$ as $n \to \infty$.

The subsequent case is derived from Case 2 in a manner analogous to that in Proposition 3. Consequently, the desired result follows. □

Theorem 2. *Let (X, d) be a complete (α, c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a generalized Kannan-type mapping with $\beta \in [0, (c+2)/(3(c+1)))$. Then T possesses a fixed point in X if T does not attain periodic points of prime period 2. Moreover, the number of fixed points of T is at most two.*

Proof. In the first showing, assume that T does not attain periodic points of prime period 2. Let $x_0 \in X$ be chosen arbitrarily. Define $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. If x_n is a fixed point of T for some $n \in \mathbb{N}$, then we have nothing to show. Let us assume that x_n is not a fixed point of T for all $n \in \mathbb{N}$. It follows that $x_0 \neq x_1$, $x_1 \neq x_2$, and so on. Also, since T does not attain periodic points of prime period 2, it follows that $x_0 \neq x_2$, $x_1 \neq x_3$, and so on. Therefore, any three consecutive elements of $\{x_n\}$ are distinct.

Now, for any $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&d(Tx_n, Tx_{n+1}) + d(Tx_{n+1}, Tx_{n+2}) + d(Tx_{n+2}, Tx_n) \\
&\quad \leqslant \beta\bigl[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1}) + d(x_{n+2}, Tx_{n+2})\bigr] \\
\implies\ &d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3}) + d(x_{n+3}, x_{n+1}) \\
&\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+3})\bigr] \qquad (6)
\end{aligned}$$

$$\begin{aligned} \implies\quad & d(x_{n+3}, x_{n+1}) \\ & \leqslant \beta d(x_n, x_{n+1}) - (1-\beta) d(x_{n+1}, x_{n+2}) - (1-\beta) d(x_{n+2}, x_{n+3}). \end{aligned} \tag{7}$$

Now, from (6), for each $n \cup \{0\}$, we have

$$\begin{aligned} &(1-\beta) d(x_{n+2}, x_{n+3}) \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] - d(x_{n+1}, x_{n+2}) - d(x_{n+3}, x_{n+1}) \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] - d(x_{n+2}, x_{n+3}) \\ &\qquad + c\bigl[d(x_{n+2}, x_{n+1})\bigr]^{\alpha}\bigl[d(x_{n+1}, x_{n+3})\bigr]^{1-\alpha}, \end{aligned}$$

which implies that

$$\begin{aligned} &(2-\beta) d(x_{n+2}, x_{n+3}) \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] + c\bigl[d(x_{n+2}, x_{n+1})\bigr]^{\alpha}\bigl[d(x_{n+1}, x_{n+3})\bigr]^{1-\alpha} \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] + c \max\bigl\{d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_{n+3})\bigr\} \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] + c \max\bigl\{d(x_{n+2}, x_{n+1}), \beta d(x_n, x_{n+1}) \\ &\qquad - (1-\beta) d(x_{n+1}, x_{n+2}) - (1-\beta) d(x_{n+2}, x_{n+3})\bigr\} \quad \text{(using (7))} \\ &\quad \leqslant \beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] + c\bigl[d(x_{n+2}, x_{n+1}) + \beta d(x_n, x_{n+1}) \\ &\qquad - (1-\beta) d(x_{n+1}, x_{n+2}) - (1-\beta) d(x_{n+2}, x_{n+3})\bigr] \\ &\quad \leqslant (c+1)\beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] - c(1-\beta) d(x_{n+2}, x_{n+3}). \end{aligned}$$

Therefore,

$$\bigl[(2-\beta) + c(1-\beta)\bigr] d(x_{n+2}, x_{n+3}) \leqslant (c+1)\beta\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr]$$

for all $n \in \mathbb{N} \cup \{0\}$. This implies that

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &\leqslant \frac{(c+1)\beta}{(2-\beta) + c(1-\beta)}\bigl[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})\bigr] \\ &\leqslant \frac{2(c+1)\beta}{(2-\beta) + c(1-\beta)} \max\bigl\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\bigr\} \end{aligned}$$

for all $n \in \mathbb{N} \cup \{0\}$. Define

$$\lambda := \frac{2(c+1)\beta}{(2-\beta) + c(1-\beta)} \in [0, 1), \quad \beta \in \biggl[0, \frac{c+2}{3(c+1)}\biggr).$$

Also, let $m_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \cup \{0\}$, and let $M := \max\{m_0, m_1\}$. Then we have

$$m_0 \leqslant M \quad \text{and} \quad m_1 \leqslant M.$$

This implies that

$$m_2 \leqslant \lambda \max\{m_0, m_1\} \leqslant \lambda M \quad \text{and} \quad m_3 \leqslant \lambda \max\{m_1, m_2\} \leqslant \lambda M.$$

By continuing this process, we can conclude that

$$m_0, m_1 \leqslant M, \quad m_2, m_3 \leqslant \lambda M, \quad m_4, m_5 \leqslant \lambda^2 M, \quad \ldots .$$

This implies that $m_n \to 0$ as $n \to \infty$, that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0.$$

Now, we have

$$\begin{aligned} d(x_n, x_{n+2}) &\leqslant d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) \\ &\quad + c\big[\big(d(x_n, x_{n+1})\big)^{\alpha}\big(d(x_{n+1}, x_{n+2})\big)^{1-\alpha}\big], \end{aligned}$$

which implies that $\lim *_{n\to\infty} d(x_n, x_{n+2}) = 0$.

Let us assume that $\lim *_{n\to\infty} d(x_n, x_{n+r}) = 0$ for some $r \in \mathbb{N}$. Then we have

$$\begin{aligned} d(x_n, x_{n+r+1}) &\leqslant d(x_n, x_{n+r}) + d(x_{n+r}, x_{n+r+1}) \\ &\quad + c\big[\big(d(x_n, x_{n+r})\big)^{\alpha}\big(d(x_{n+r}, x_{n+r+1})\big)^{1-\alpha}\big], \end{aligned}$$

which implies that $\lim *_{n\to\infty} d(x_n, x_{n+r+1}) = 0$. Inductively, we have

$$\lim_{n\to\infty} d(x_n, x_{n+r}) = 0 \quad \text{for arbitrary but fixed } r \in \mathbb{N}.$$

Now, following the same line as in the proof of Theorem 1, it can be shown that

$$\lim_{n\to\infty} \sup\big\{d(x_n, x_m)\colon m > n\big\} = 0.$$

Hence, $\{x_n\}$ is a Cauchy sequence in X. Now, by the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$.

Observe that any three consecutive elements of the sequence $\{x_n\}$ are pairwise distinct. Assume that $x^* \neq x_k$ for all $k \in \{1, 2, \ldots\}$. In this case, (5) applies to the distinct points x^*, x_{n-1}, and x_n for all $n \in \mathbb{N}$. Suppose now that there exists a smallest index $k \in \{1, 2, \ldots\}$ such that $x^* = x_k$. Let $m > k$ be such that $x^* = x_m$ again. Then the sequence $\{x_n\}$ becomes eventually periodic starting from index k, which prevents it from being a Cauchy sequence. Consequently, the elements x^*, x_{n-1}, and x_n remain distinct for all n sufficiently large such that $n - 1 > k$. Now, we have

$$\begin{aligned} d(x_n, Tx^*) &= d(Tx_{n-1}, Tx^*) \\ &\leqslant d(Tx_{n-1}, Tx^*) + d(Tx^*, Tx_n) + d(Tx_n, Tx_{n-1}) \\ &\leqslant \beta\big[d(x_{n-1}, Tx_{n-1}) + d(x^*, Tx^*) + d(x_n, Tx_n)\big] \\ &\leqslant \beta\big[d(x_{n-1}, x_n) + d(x^*, Tx^*) + d(x_n, x_{n+1})\big]. \end{aligned}$$

Taking the limit $n \to \infty$, we get

$$(1-\beta) d(x^*, Tx^*) \leqslant 0 \quad \Longrightarrow \quad d(x^*, Tx^*) = 0 \quad \Longrightarrow \quad Tx^* = x^*.$$

Therefore, $x^* \in \mathrm{Fix}(T)$.

Finally, let us suppose that T has three distinct fixed points, say, a, b, and c. Then $Ta = a$, $Tb = b$, and $Tc = c$. Thus, from (5) we have

$$d(a,b) + d(b,c) + d(c,a) = 0,$$

which contradicts the distinctness of a, b, and c. Hence, T has at most two fixed points. This completes the proof. □

As a particular case, when $c = 0$, the following result of [18, p. 7, Thm. 3.2] follows from Theorem 2.

Corollary 2. *Let (X,d), $|X| \geqslant 3$, be a complete metric space, and let the mapping $T : X \to X$ satisfy the following two conditions*:

(i) *T does not possess periodic points of prime period 2;*
(ii) *T is a generalized Kannan-type mapping on X.*

Then T has a fixed point. Moreover, the number of fixed points is at most two.

Interestingly, the converse of the above corollary can indeed be established within the setting of metric spaces. In contrast, its validity within interpolative metric spaces remains an open question.

Theorem 3. *Let (X,d) be a complete metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a generalized Kannan-type mapping on X. If T has a fixed point, then T does not possess periodic points of prime period 2.*

Proof. Suppose that T have a fixed point, say x. For contradiction, assume that T admits a periodic point y of prime period 2, and let $Ty = z$. Then from (1) we have

$$\begin{aligned}
&d(Tx,Ty) + d(Ty,Tz) + d(Tz,Tx) \leqslant \beta\big[d(x,Tx) + d(y,Ty) + d(z,Tz)\big]\\
&\quad\Longrightarrow\quad d(x,z) + d(z,y) + d(y,x) \leqslant \beta\big[d(x,x) + d(y,z) + d(z,y)\big]\\
&\quad\Longrightarrow\quad 2d(y,z) \leqslant d(x,z) + d(z,y) + d(y,x) \leqslant 2\beta d(y,z)\\
&\quad\Longrightarrow\quad \beta \geqslant 1.
\end{aligned}$$

This contradicts (1), and hence the result follows. □

Open problem. *The converse of Theorem 2 remains open to researchers to explore.*

The following example is provided to demonstrate and substantiate the validity of Theorem 2.

Example 8. Let $X = [0,10]$, and let d be the Euclidean metric. Consider $\mathcal{D} : X \times X \to [0,\infty)$ defined as

$$\mathcal{D}(x,y) = d(x,y)\big[d(x,y) + 1\big] \quad \text{for all } x,y \in X.$$

Then $(X,\mathcal{D})$ is a $(1/2,2)$-interpolative metric space.

Suppose that $T : X \to X$ is defined by

$$Tx = \begin{cases} 0 & \text{if } x \in [0, 5], \\ 1 & \text{if } x \in (5, 10]. \end{cases}$$

Without loss of generality, suppose that $x < y < z$, and let

$$U(x, y, z) = d(Tx, Ty) + d(Ty, Tz) + d(Tz, Tx),$$
$$V(x, y, z) = d(x, Tx) + d(y, Ty) + d(z, Tz).$$

Now, if $x < y < z \leqslant 5$, then $U(x, y, z) = 0$.

For $x < y \leqslant 5 \leqslant z$, we get

$$\begin{aligned} U(x, y, z) &= 4, \\ V(x, y, z) &= |x|\bigl(|x| + 1\bigr) + |y|\bigl(|y| + 1\bigr) + |z - 1|\bigl(|z - 1| + 1\bigr) \\ &\geqslant 20. \end{aligned}$$

For $x \leqslant 5 < y < z$, we get

$$\begin{aligned} U(x, y, z) &= 4, \\ V(x, y, z) &= |x|\bigl(|x| + 1\bigr) + |y - 1|\bigl(|y - 1| + 1\bigr) + |z - 1|\bigl(|z - 1| + 1\bigr) \\ &\geqslant 40. \end{aligned}$$

Now, if $5 < x < y < z$, then $U(x, y, z) = 0$.

In all cases, we get $U(x, y, z) \leqslant V(x, y, z)/5$ for all distinct $x, y, z \in X$. Then T is a generalized Kannan-type mapping with $\beta \in [1/5, 4/9)$, and T does not consist of periodic points of prime period 2. Thus, Theorem 2 guarantees that T has a fixed point. Clearly, $\mathrm{Fix}(T) = \{0\}$.

Next, we present an example of a generalized Kannan-type mapping with exactly two fixed points.

Example 9. Let $X = \{p, q, r\}$, and let $d : X \times X \to [0, \infty)$ be defined by

$$d(p, q) = 3, \qquad d(q, r) = 8, \qquad d(p, r) = 15.$$

Then d is not a metric, but d is a $(1/2, 2)$-interpolative metric on X.

Let $T : X \to X$ be defined as $Tp = p$, $Tq = q$, and $Tr = p$. Then T is a generalized Kannan-type mapping with $\beta \in [3/13, 4/9)$, and T does not contain periodic points of prime period 2. Thus, Theorem 2 guarantees the existence of a fixed point of T. Clearly, $\mathrm{Fix}(T) = \{p, q\}$.

We aim to demonstrate that none of the conditions outlined in Theorem 2 can be weakened concerning the existence of fixed points. To begin, we present Example 10, which illustrates that the absence of periodic points of prime period 2 is essential for the existence of fixed points. Example 11 show the necessity of the contraction condition.

Example 10. Let $X = \{0, 1, 2, 3\}$, and let $\delta : X \times X \to [0, \infty)$ be defined by

$$\delta(x, y) = |x - y|^3 \quad \text{for all } x, y \in X.$$

Then (X, δ) is a $(1/3, 6)$-interpolative metric space. Let $T : X \to X$ be defined by $T(0) = 1$, $T(1) = 0$, $T(2) = 1$, and $T(3) = 1$. Then T is a generalized Kannan-type mapping with $\beta \in [2/11, 8/21)$. Hence, $0, 1$ are the periodic points of prime period 2. Notice that T has no fixed point.

Example 11. Let $X = \{d, e, f\}$ be endowed with an (α, c)-interpolative metric ρ, and let $T : X \to X$ be defined by $Td = f$, $Te = d$, and $Tf = e$. Then we have

$$\rho(Td, Te) + \rho(Te, Tf) + \rho(Tf, Td) = \rho(d, f) + \rho(e, d) + \rho(f, e).$$

Thus, (5) does not hold, and hence, T is not a generalized Kannan-type mapping. Although T does not contain any periodic points of prime period 2, it does not have a fixed point.

It is obvious from Theorem 2 that the fixed points of the generalized Kannan-type mapping may not be unique. On the other hand, we can create a situation where the uniqueness of the fixed point is guaranteed. The proof of the following result is similar to that of the metric space, so it is omitted here.

Proposition 5. *Let (X, d) be a complete (α, c)-interpolative metric space with $|X| \geqslant 3$, and let $T : X \to X$ be a generalized Kannan-type mapping with no periodic point of prime period 2. For $x_0 \in X$, if the infinite iterative sequence x_0, $x_1 = Tx_0$, $x_2 = Tx_1$, $\dots$, converges to a point $\xi \in X$ with $\xi \neq x_n$ for all $n \in \mathbb{N} \cup \{0\}$, then ξ is the unique fixed point of T.*

4 Conclusions

In this paper, mappings that contract the perimeter of triangles are investigated in the setting of an interpolative metric space. A necessary and sufficient condition is obtained for the existence of fixed points of these mappings. Ensuring the absence of periodic points of prime period 2 is essential for achieving a fixed point. These mappings can have at most two fixed points. Thus, a necessary condition is derived for the uniqueness of the fixed point. Furthermore, generalized Kannan-type mappings are examined within the framework of interpolative metric spaces. A necessary condition is established for the existence of fixed points of these mappings, and a fixed point result of these mappings in metric spaces naturally emerges as a corollary of our result. Using our result, the Banach contraction principle is proved in the setting of an interpolative metric space in an alternative way. This new inequality condition in the interpolative metric space can be highly beneficial for calculations and obtaining more accurate approximations of fixed points. Furthermore, nontrivial examples are presented to validate our findings.

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