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The asymptotic behavior of systemic expected shortfall and marginal expected shortfall under extreme losses*

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Abstract. The paper considers the asymptotics of the systemic expected shortfall (SES) and the marginal expected shortfall (MES) in systemic risks. We mainly consider individual losses to be the products of primary variables and random weights, where a dependence structure exists between the primary variable and the corresponding random weight for each loss. When the distributions of the primary variables belong to the intersection of long-tailed and dominatedly varying-tailed distribution function classes, we obtain the asymptotics of SES and MES of individual losses. Particularly, under the coherent capital allocation, more precise estimations for SES and MES of individual losses are presented for the regularly varying-tailed primary variables. Some numerical studies are conducted to check the accuracy of the obtained results.

Keywords: systemic risk, systemic expected shortfall, marginal expected shortfall, heavy-tailed distribution function.

1 Introduction

Systemic risks refer to the changes and impacts of various relevant factors, which may increase the risk exposure of investors and result in corresponding losses. In recent decades, systemic risks have played an important role in interpreting the financial turbulence of insurance and financial fields, especially in banks and financial industries. It has also become an important research content. Scholars have proposed various measures on systemic risks. One can refer to [2, 7, 13] and so on. Acharya et al. [1] presented the concepts of the systemic expected shortfall (SES) and the marginal expected shortfall (MES). The SES refers to the total expected loss collectively incurred by all affected financial

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institutions. The MES, meanwhile, indicates the exposure degree of a specific asset or portfolio to systemic risks. In the insurance field, there also have some investigations on systemic risks. For related references, see [5, 14, 24] and so on.

In the studies on risk measures, [4] and [22] proposed an asymptotic approach about the estimations of systemic risks measures. Recently, [8] derived the asymptotic results of the SES and MES under the case of heavy-tailed losses. On the basis of [4, 7, 8, 22], this paper aims to further estimate the asymptotics of the SES and MES of a system under extreme losses. As [1,8], we consider a static one-period model in which a system consists n ($n \geqslant 1$) individuals (or subportfolios within the investment portfolios or business lines or companies), generating loss-profit variables $Z_1, \dots, Z_n$, respectively. Thus, the system has an aggregate loss $S_n = \sum_{i=1}^n Z_i$. Suppose the system follows the internal model approach and use a risk measure ρ to determine the solvency capital requirement. The solvency capital requirement of the i ($i = 1, 2, \dots, n$) individual is A_i satisfying $\rho(S_n) = A = \sum_{i=1}^n A_i$. For an individual $k = 1, 2, \dots, n$, its SES and MES are defined as, respectively,

$$\mathrm{SES}_k = \mathbf{E}\big[(Z_k - A_k)^+ \bigm| S_n > A\big]$$

and

$$\mathrm{MES}_k = \mathbf{E}\big[Z_k \bigm| S_n > A\big],$$

where $a^+ = a\mathbf{1}_{\{\mathbf{a} \geqslant \mathbf{0}\}}$ represents the positive part of a real number a, and $\mathbf{1}_{\{\mathbf{A}\}}$ represents the indicator of an event A. As with the coherent capital allocation rule introduced by [19], the expected shortfall is associated with a confidence level $q \in (0, 1)$, which makes the solvency capital A determined by the expected shortfall and the solvency capital A_i, $i = 1, 2, \dots, n$ allocated to each individual being related to the confidence level q. Thus, we assume that A and A_i ,$i = 1, 2, \dots, n$, are functions of the confidence level $q \in (0, 1)$. To reflect this relation, we rewrite the definitions of SES and MES for an individual $k = 1, 2, \dots, n$ as

$$\mathrm{SES}_k(q) = \mathbf{E}\big[\big(Z_k - A_k(q)\big)^+ \bigm| S_n > A(q)\big] \tag{1}$$

and

$$\mathrm{MES}_k(q) = \mathbf{E}\big[Z_k \bigm| S_n > A(q)\big]. \tag{2}$$

Under the EU Solvency II Directive 2009/138/EC, capital requirements are determined on the basis of a 99.5% VaR measure, that is, insurance companies must have sufficient capital to cover losses due to change in market value over the next year with a confidence level of 99.5%. The requirement for such a high confidence level makes it important to study the risk when the confidence level $q \uparrow 1$, especially under extreme risks. Hence, for each $k = 1, 2, \dots, n$, this paper will discuss the asymptotics of $\mathrm{SES}_k(q)$ and $\mathrm{MES}_k(q)$ when $q \uparrow 1$. Throughout the paper, for any $k = 1, 2, \dots, n$, we assume that $A_k(q) \to \infty$ as $q \uparrow 1$.

This paper will use a popular framework with random weights to model individual losses $Z_1, \dots, Z_n$. For each $i = 1, 2, \dots, n$, the loss-profit variable of the ith individual has the following form $Z_i = X_i\theta_i$, where X_i is a primary variable, controlling the tail behavior of Z_i, and θ_i is a random weight, reflecting the degree of impact of some economic factors to the ith individual. In this paper, we assume that $X_1, \dots, X_n$ are n real-valued random variables with distribution functions $F_1, \dots, F_n$, respectively, and $\theta_1, \dots, \theta_n$ are n nonnegative nondegenerate at zero random weights with distribution functions $G_1, \dots, G_n$, respectively. Then $S_n = \sum_{i=1}^{n} X_i\theta_i$ is a randomly weighted sum.

We will use the method of processing randomly weighted sums to study the asymptotics of SES and MES. In the study of randomly weighted sum, one aspect discusses the case where the primary random variables and the random weights are independent, such as [9, 11, 18, 26–28] and so on. Another aspect investigates the scenario where a dependence structure exists between the primary random variables and the random weights, as seen in [29] and [31]. For the first case, [8] considered the case where the primary variables $\{X_i,\, i = 1, 2, \dots, n\}$ have a strongly quasi-asymptotic independence structure, and the random weights $\{\theta_i,\, i = 1, 2, \dots, n\}$ are arbitrarily dependent and independent of $\{X_i,\, i = 1, 2, \dots, n\}$. They gave the asymptotic properties of SES and MES. [30] also considered random weights $\{\theta_i,\, i = 1, 2, \dots, n\}$ independent of the primary variables $\{X_i,\, i = 1, 2, \dots, n\}$. They investigated the asymptotics of SES and MES for the case where the primary variables $\{X_i,\, i = 1, 2, \dots, n\}$ are conditionally linearly wide dependent or have a dependence structure introduced by [20], and the random weights $\{\theta_i,\, i = 1, 2, \dots, n\}$ are bound. This paper, in contrast, examines the second case, which involves a dependence structure between the primary variable X_i and the random weight θ_i for each $i = 1, 2, \dots, n$.

In the following, we will introduce a dependence structure, which was first proposed by [3] and then redefined and extended by [10] and [21].

Definition 1. Let ξ be a real-valued and unbounded random variable, and let η be a nonnegative random variable. Say that the random pair (ξ, η) satisfies the dependence structure $\mathscr{H}$ if there is some nonnegative function $h(\cdot)$, which is bounded away from zero in a left neighborhood $\varDelta$ of x_η such that

$$\frac{\mathbf{P}(\xi > x | \eta = y)}{\mathbf{P}(\xi > x)} \to h(y), \quad x \to \infty \tag{3}$$

holds uniformly for all $y \in R(\eta)$, where $R(\eta) = \{\eta(\omega),\, \omega \in \varOmega\}$, consisting of all possible values of η, $\varOmega$ is the sample space, and $x_\eta = \sup\{y\colon \mathbf{P}(\eta \leqslant y) < 1\}$ is the upper endpoint of η. If $y \notin R(\eta)$, the conditional probability is understood as unconditional, and, therefore, $h(y) = 1$, $y \geqslant 0$. The left neighborhood $\varDelta$ of x_η is understood as $(x_\eta - \varepsilon, x_\eta]$ for some small $\varepsilon > 0$ if $x_\eta < \infty$, and as (D, ∞) for some large $D > 0$ if $x_\eta = \infty$.

Integrating both sides of (3) with respect to $\mathbf{P}(\eta \in \mathrm{d}y)$ leads to $\mathbf{E}[h(\eta)] = 1$. This paper will consider each pair of primary random variable and random weight (X_i, θ_i), $i = 1, 2, \dots, n$, has a dependence structure $\mathscr{H}$. A detailed assumption is the following.

Assumption 1. Let (X_i, θ_i), $i = 1, 2, \dots, n$, be n independent random vectors. For each $i = 1, 2, \dots, n$, (X_i, θ_i) satisfies the dependence structure $\mathscr{H}$, i.e., there exists a nonnegative function $h_i(\cdot)$ such that (3) holds uniformly for all $y \in R(\theta_i)$.

Under Assumption 1, for each $k = 1, 2, \dots, n$, we will investigate the asymptotics of $\mathrm{SES}_k(q)$ and $\mathrm{MES}_k(q)$ as $q \uparrow 1$ for the heavy-tailed primary variables. The remainder of this paper is organized into four sections. Section 2 presents some preliminaries on the heavy-tailed distribution functions. Section 3 gives the main results of the paper, and their proofs will be given in Section 5. Section 4 contains some numerical studies to examine the accuracy of our asymptotic results.

2 Preliminaries

For two positive functions $f(x)$ and $g(x)$, we write

$$\begin{aligned}
&f(x) \underset{x\to\infty}{=} o\big(g(x)\big) \quad \text{if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 0,\\
&f(x) \underset{x\to\infty}{=} O\big(g(x)\big) \quad \text{if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty,\\
&f(x) \underset{x\to\infty}{\lesssim} g(x) \ \text{ or } \ g(x) \underset{x\to\infty}{\gtrsim} f(x) \quad \text{if } \limsup_{x\to\infty} \frac{f(x)}{g(x)} \leqslant 1,\\
&f(x) \underset{x\to\infty}{\sim} g(x) \quad \text{if } \lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,\\
&f(x) \underset{x\to\infty}{\asymp} g(x) \quad \text{if } 0 < \liminf_{x\to\infty} \frac{f(x)}{g(x)} \leqslant \limsup_{x\to\infty} \frac{f(x)}{g(x)} < \infty.
\end{aligned}$$

For two real number x and y, let $x \vee y = \max\{x, y\}$. For any $-\infty < a \leqslant b < \infty$, the integral symbol $\int_a^b$ is understood as $\int_{(a,b]}$. When $b = \infty$, the integral is interpreted over the interval (a, ∞).

We first recall the definition of a heavy-tailed distribution function and several important subclasses of its class. In the following, let V be a distribution function having a right tail that does not vanish in the sense that $\overline{V}(x) = 1 - V(x) > 0$ for all $x \in (-\infty, \infty)$. A distribution function V on $(-\infty, \infty)$ is said to be heavy-tailed if for any $\lambda > 0$, $\int_{-\infty}^{\infty} \mathrm{e}^{\lambda x}\, V(\mathrm{d}x) = \infty$. A distribution function V on $[0, \infty)$ is said to be subexponential, denoted by $V \in \mathscr{S}$, if

$$\lim_{x\to\infty} \frac{\overline{V^{*n}}(x)}{\overline{V}(x)} = n$$

for some (or, equivalently, for all) $n \geqslant 2$, where V^{*n} denotes the n-fold convolution of V, $n \geqslant 2$. Further, say that a distribution function V on $(-\infty, \infty)$ belongs to the class $\mathscr{S}$ if $V(x)\mathbf{1}_{\{x\geqslant 0\}}$ belongs to the class $\mathscr{S}$. If for any $u > 0$,

$$\lim_{x\to\infty} \frac{\overline{V}(x+u)}{\overline{V}(x)} = 1,$$

then V is said to be long-tailed, denoted by $V \in \mathscr{L}$. If for any $0 < u < 1$,

$$\limsup_{x\to\infty} \frac{\overline{V}(xu)}{\overline{V}(x)} < \infty,$$

then V is said to be dominatedly varying-tailed, denoted by $V \in \mathscr{D}$. If for any $u > 0$ and some $0 \leqslant \alpha < \infty$,

$$\lim_{x\to\infty} \frac{\overline{V}(xu)}{\overline{V}(x)} = u^{-\alpha},$$

then V is said to be regularly varying-tailed, denoted by $V \in \mathscr{R}_{-\alpha}$. Let

$$\mathscr{R} = \bigcup_{\alpha \geqslant 0} \mathscr{R}_{-\alpha}.$$

The above distribution classes have the following relations:

$$\mathscr{R} \subset \mathscr{L} \cap \mathscr{D} \subset \mathscr{S} \subset \mathscr{L};$$

see, e.g. [12, 15, 16].

For a distribution function V on $(-\infty, \infty)$, define its lower and upper Matuszewska indices, respectively, by

$$J_V^- = \sup\left\{-\frac{\log \overline{V}^*(y)}{\log y}, \, y \geqslant 1\right\} \quad \text{and} \quad J_V^+ = \inf\left\{-\frac{\log \overline{V}_*(y)}{\log y}, \, y \geqslant 1\right\},$$

where

$$\overline{V}^*(y) = \limsup_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)} \quad \text{and} \quad \overline{V}_*(y) = \liminf_{x\to\infty} \frac{\overline{V}(xy)}{\overline{V}(x)}, \quad y \geqslant 0.$$

Set $L_V = \lim_{y \searrow 1} \overline{V}_*(y)$.

3 Main results

In the following, we will present the asymptotic behavior of SES and MES.

Theorem 1. *Consider* SES *and* MES *in* (1) *and* (2). *Assume that* (X_i, θ_i), $i = 1, 2, \dots, n$, *satisfies Assumption* 1. *Suppose that for all* $i = 1, 2, \dots, n$, $F_i \in \mathscr{L} \cap \mathscr{D}$, $J_{F_i}^- > 1$, $\overline{G}_i(x) \underset{x\to\infty}{=} o(\overline{F}_i(x))$, *and* $\mathbf{E}[h_i(\theta_i)\theta_i^{p_1}] < \infty$ *for some* $p_1 > \vee_{i=1}^n J_{F_i}^+$. *If there exists an individual* k $(k = 1, 2, \dots, n)$ *such that* $\overline{F}_i(x) \underset{x\to\infty}{=} O(\overline{F}_k(x))$, $1 \leqslant i \neq k \leqslant n$, *and* $A(q) \underset{q\uparrow 1}{=} O(A_k(q))$, *then*

$$\begin{aligned}\mathrm{SES}_k(q) &\underset{q\uparrow 1}{\sim} \frac{\mathbf{P}(Z_k > A(q)) \sum_{i=1,\, i\neq k}^n A_i(q) + \mathbf{E}[(Z_k - A(q))^+]}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))} \\ &= \frac{\mathbf{E}[Z_k \mathbf{1}_{\{Z_k > A(q)\}}] - A_k(q)\mathbf{P}(Z_k > A(q))}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))}.\end{aligned}$$

Furthermore, if $\mathbf{E}[|Z_k|] < \infty$, $\mathbf{E}[\theta_k^{p_2}] < \infty$ *for some* $p_2 > J_{F_k}^+$, *and there exist positive constants* $\widetilde{D}_k$ *and* $\widetilde{C}_k$ *such that for all* $x \geqslant \widetilde{D}_k$ *and* $y \in R(\theta_k)$,

$$\mathbf{P}(X_k < -x \mid \theta_k = y) \leqslant \widetilde{C}_k \mathbf{P}(X_k > x \mid \theta_k = y), \tag{4}$$

then

$$\mathrm{MES}_k(q) \underset{q\uparrow 1}{\sim} \frac{\mathbf{E}[Z_k \mathbf{1}_{\{Z_k > A(q)\}}]}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))}.$$

In the following, we consider a coherent capital allocation rule, which is introduced by [19]. The concrete estimations for $\mathrm{SES}_k(q)$ and $\mathrm{MES}_k(q)$ can be given. For a general risk variable ξ with a distribution function V, its value at risk (VaR) and expected shortfall (ES) at level $0 < q < 1$ are defined by, respectively,

$$\mathrm{VaR}_q(\xi) = V^{\leftarrow}(q) = \inf\{x \in (-\infty, \infty)\colon V(x) \geqslant q\}$$

and

$$\mathrm{ES}_q(\xi) = \frac{1}{1-q} \int_q^1 \mathrm{VaR}_p(\xi)\,\mathrm{d}p.$$

According to Section 5.2 of [19], under $\rho(S_n) = \mathrm{ES}_q(S_n)$, the coherent capital allocation rule assigns to each $k = 1, 2, \dots, n$

$$A_k = \frac{\mathbf{P}(S_n > H_n^{\leftarrow}(q))}{1-q} \mathbf{E}\bigl[Z_k \bigm| S_n > H_n^{\leftarrow}(q)\bigr] + \frac{\mathbf{P}(S_n \leqslant H_n^{\leftarrow}(q)) - q}{1-q} \mathbf{E}\bigl[Z_k \bigm| S_n = H_n^{\leftarrow}(q)\bigr],$$

where H_n is the distribution function of S_n.

Corollary 1. *Consider* SES *and* MES *in* (1) *and* (2) *under the coherent capital allocation rule. Assume that* (X_i, θ_i), $i = 1, 2, \dots, n$, *satisfy Assumption* 1 *and there is a distribution function* F *on* $(-\infty, \infty)$ *satisfying* $F \in \mathscr{R}_{-\alpha}$ *for some* $\alpha > 1$ *and* $\overline{F}_i(x) \underset{x\to\infty}{\sim} b_i \overline{F}(x)$ *for all* $i = 1, 2, \dots, n$ *and a constant* $b_i > 0$. *If for all* $i = 1, 2, \dots, n$, $\overline{G}_i(x) \underset{x\to\infty}{=} o(\overline{F}_i(x))$ *and* $\mathbf{E}[h_i(\theta_i)\theta_i^{p_1}] < \infty$ *for some* $p_1 > \alpha$, *then for each* $k = 1, 2, \dots, n$,

$$\mathrm{SES}_k(q) \underset{q\uparrow 1}{\sim} C_n^{1/\alpha - 1} \frac{\alpha}{\alpha - 1} b_k \mathbf{E}\bigl[h_k(\theta_k)\theta_k^{\alpha} \mathbf{1}_{\{\theta_k > 0\}}\bigr] \times \left[C_n^{-1} \sum_{i=1,\, i\neq k}^{n} b_i \mathbf{E}\bigl[h_i(\theta_i)\theta_i^{\alpha} \mathbf{1}_{\{\theta_i > 0\}}\bigr] + \frac{1}{\alpha - 1}\right] F^{\leftarrow}(q),$$

where

$$C_n = \sum_{i=1}^n b_i \mathbf{E}\bigl[h_i(\theta_i)\theta_i^{\alpha} \mathbf{1}_{\{\theta_i > 0\}}\bigr].$$

Furthermore, if for some $k = 1, 2, \dots, n$, (X_k, θ_k) *satisfies* (4), $\mathbf{E}[|Z_k|] < \infty$, *and* $\mathbf{E}[\theta_k^{p_2}] < \infty$ *for some* $p_2 > \alpha$, *then*

$$\mathrm{MES}_k(q) \underset{q\uparrow 1}{\sim} b_k \mathbf{E}\bigl[h_k(\theta_k)\theta_k^{\alpha} \mathbf{1}_{\{\theta_k > 0\}}\bigr] C_n^{1/\alpha - 1} \left(\frac{\alpha}{\alpha - 1}\right)^2 F^{\leftarrow}(q).$$

4 Numerical studies

Based on the results in Corollary 1, we will verify the accuracy of the asymptotic results in this section. Assume that there are two lines of business with the ith ($i = 1, 2$) line generating loss X_i that follows a Pareto distribution function

$$F_i(x) = 1 - \left(\frac{i}{x+i}\right)^{\alpha}, \quad x \geqslant 0,$$

where $\alpha > 0$ is a shape parameter. For each $i = 1, 2$, suppose that the joint distribution function of $(X_i, theta_i)$ is a Farlie–Gumbel–Morgenstern (FGM) distribution function

$$\pi(x, y) = F_i(x)G_i(y)\bigl(1 + \gamma\overline{F}_i(x)\overline{G}_i(y)\bigr), \quad x, y \geqslant 0,$$

and the corresponding copula has the following form:

$$C(u, v) = uv\bigl(1 + \gamma(1-u)(1-v)\bigr), \quad u, v \in [0, 1],$$

where $\gamma \in [-1, 1]$ is a dependence coefficient. As noted in [21], the FGM copula satisfies the dependence structure $\mathscr{H}$, and for each $i = 1, 2$, $h_i(y) = 1 + \gamma(1 - 2\overline{G}_i(y))$, $y \geqslant 0$.

Furthermore, we assume that θ_i, $i = 1, 2$, follow a common exponential distribution with parameter 2. Generate $N = 10^7$ samples $(\theta_{1,j}, \theta_{2,j}; X_{1,j}, X_{2,j})$, $j = 1, \dots, N$, and denote $S_j = \sum_{i=1}^{2} \theta_{i,j}X_{i,j}$, $j = 1, \dots, N$, with $S_{(1)} \leqslant \dots \leqslant S_{(N)}$. Then the empirical estimator for $A_i(q)$ is given by

$$\widehat{A}_i(q) = \frac{\sum_{j=1}^{N} \theta_{i,j}X_{i,j}\mathbf{1}_{\{S_j > S_{(\lfloor N_q \rfloor)}\}}}{\sum_{j=1}^{N} \mathbf{1}_{\{S_j > S_{(\lfloor N_q \rfloor)}\}}}, \quad i = 1, 2,$$

and $\widehat{A}(q) = \widehat{A}_1(q) + \widehat{A}_2(q)$. The empirical estimates for $\mathrm{SES}_k(q)$ and $\mathrm{MES}_k(q)$ are generated by

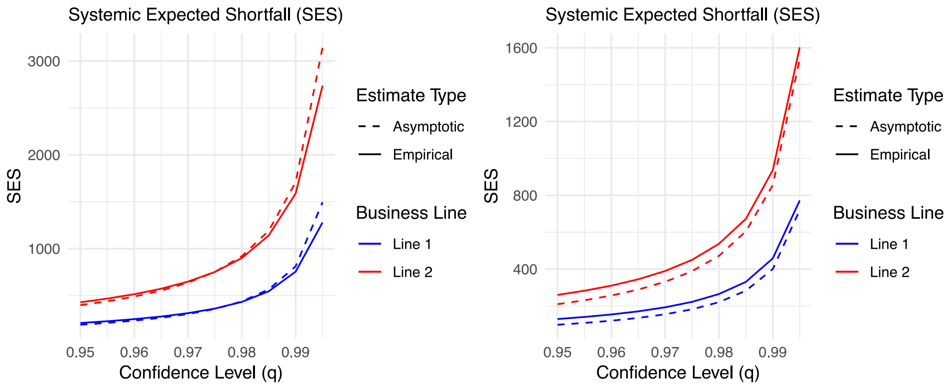
$$\widehat{\mathrm{SES}}_k(q) = \frac{\sum_{j=1}^{N} \bigl(\theta_{k,j}X_{k,j} - \widehat{A}_k(q)\bigr)^{+}\mathbf{1}_{\{S_j > \widehat{A}(q)\}}}{\sum_{j=1}^{N} \mathbf{1}_{\{S_j > \widehat{A}(q)\}}}, \quad k = 1, 2,$$

and

$$\widehat{\mathrm{MES}}_k(q) = \frac{\sum_{j=1}^{N} \theta_{k,j}X_{k,j}\mathbf{1}_{\{S_j > \widehat{A}(q)\}}}{\sum_{j=1}^{N} \mathbf{1}_{\{S_j > \widehat{A}(q)\}}}, \quad k = 1, 2.$$

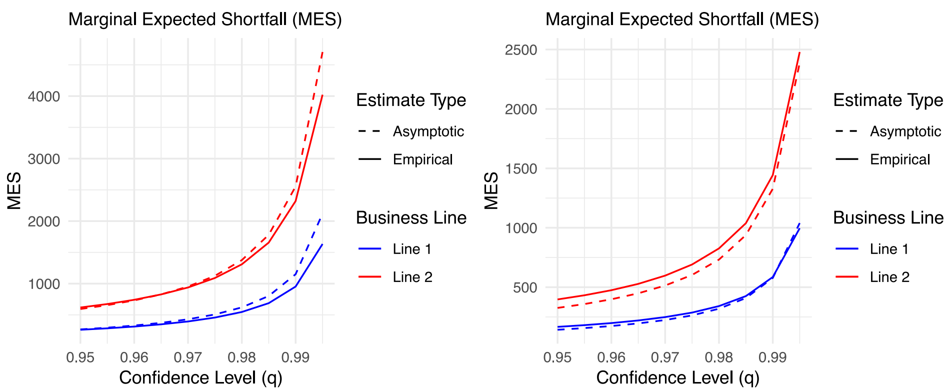
The asymptotic results follow directly from Corollary 1. For dependence coefficients $\gamma = 0.4$ and $\gamma = 0.2$, we vary q from 0.95 to 1 in increments of 0.005. The resulting Figs. 1(a) and 1(b) correspond to $\gamma = 0.4$ with $\alpha = 1.15$ and $\alpha = 1.2$, respectively, whereas Figs. 2(a) and 2(b) correspond to $\gamma = 0.2$ with $\alpha = 1.15$ and $\alpha = 1.2$, respectively.

From Figs. 1–2, although the shape parameter α is different, the values of $\mathrm{SES}_1(q)$, $\mathrm{SES}_2(q)$, $\mathrm{MES}_1(q)$, and $\mathrm{MES}_2(q)$ all increase as $q \uparrow 1$. The asymptotic estimates for both SES and MES show a good fit with their empirical counterparts, thereby verifying the accuracy of Corollary 1.



(a) $\alpha = 1.15$ (b) $\alpha = 1.2$

Figure 1. Comparisons between the asymptotic estimate and empirical estimate of SES.



(a) $\alpha = 1.15$ (b) $\alpha = 1.2$

Figure 2. Comparisons between the asymptotic estimate and empirical estimate of MES.

5 Proofs of main results

5.1 Some lemmas

Before giving the proofs of the main results, we first present some lemmas. The first lemma is a combination of Proposition 2.2.1 of [6] and Lemma 3.5 of [25].

Lemma 1. *Let V be a distribution function on $(-\infty, \infty)$. If $V \in \mathscr{D}$, then for each $0 < \widetilde{p} < J_V^- \leqslant J_V^+ < p$, there exist positive constants C_i and D_i, $i = 1, 2$, such that*

$$\frac{\overline{V}(y)}{\overline{V}(x)} \leqslant C_1 \left(\frac{y}{x}\right)^{-p}, \quad x \geqslant y \geqslant D_1; \qquad \frac{\overline{V}(y)}{\overline{V}(x)} \geqslant C_2 \left(\frac{y}{x}\right)^{-\widetilde{p}}, \quad x \geqslant y \geqslant D_2.$$

Furthermore, for any $p > J_V^+$, it holds that

$$x^{-p} \underset{x\to\infty}{=} o\big(\overline{V}(x)\big).$$

Lemma 2. *Let ξ be a real-valued random variable with a distribution function V_1, and let η be a nonnegative nondegenerate at zero random variable with a distribution function V_2. Assume that the random pair (ξ,η) satisfies the dependence structure $\mathscr{H}$ with a nonnegative function $h(\cdot)$.*

(i) *If $V_1 \in \mathscr{D}$, then*

$$\liminf_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} \geqslant \mathbf{E}\left[h(\eta)\overline{V}_{1*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>\mathbf{0}\}}\right] > \mathbf{0}.$$

(ii) *If $V_1 \in \mathscr{D}$, $\overline{V}_2(x) \underset{x\to\infty}{=} o(\overline{V}_1(x))$, and $\mathbf{E}[h(\eta)\eta^p] < \infty$ for some $p > J_{V_1}^+$, then the distribution function of $\xi\eta$ belongs to the class $\mathscr{D}$, and*

$$\limsup_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} \leqslant \mathbf{E}\left[h(\eta)\overline{V}_1^*\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] < \infty.$$

(iii) *If $V_1 \in \mathscr{L}$ and $\overline{V}_2(x) \underset{x\to\infty}{=} o(\overline{V}_1(cx))$ for some $c > 0$, then the distribution function of $\xi\eta$ belongs to the class $\mathscr{L}$.*

Remark 1.

(i) If $V_1 \in \mathscr{R}_{-\alpha}$ for some $\alpha \geqslant 0$, then for any $y > 0$, $\overline{V}_1^*(y) = \overline{V}_{1*}(y) = y^{-\alpha}$. Therefore, it follows from Lemma 2 that

$$\lim_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} = \mathbf{E}\big[h(\eta)\eta^{\alpha}\mathbf{1}_{\{\eta>0\}}\big],$$

which is Lemma 2.4 of [31].

(ii) If ξ and η are independent, then $h(y) = 1$, $y \geqslant 0$. Since $\mathbf{E}[\eta^p] < \infty$ for some $p > J_{V_1}^+$, there exists $q \in (J_{V_1}^+, p)$ such that $x^q\overline{V}_2(x) \to 0$ and $x^q\overline{V}_1(x) \to \infty$ as $x \to \infty$. Hence, $\overline{V}_2(x) \underset{x\to\infty}{=} o(\overline{V}_1(x))$, which means that the condition $\overline{V}_2(x) \underset{x\to\infty}{=} o(\overline{V}_1(x))$ is redundant when ξ and η are independent. In this case, it follows from Lemma 2 that

$$\begin{aligned} 0 &< \mathbf{E}\left[\overline{V}_{1*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] \leqslant \liminf_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} \\ &\leqslant \limsup_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} \leqslant \mathbf{E}\left[\overline{V}_1^*\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] \\ &< \infty, \end{aligned}$$

which is Theorem 3.3(iv) of [12].

Proof of Lemma 2. Result (iii) is exactly Lemma 2.1 of [31]. We only need to prove (i) and (ii).

(i) We will first show that there is a left neighborhood Δ of x_η such that $\inf_{y\in\Delta} h(y) \in (0,\infty)$ and $\mathbf{P}(\eta \in \Delta) > 0$. Here $\Delta = (\varepsilon_0, x_\eta]$ for some $\varepsilon_0 \in (0, x_\eta)$ if $x_\eta < \infty$ and $\Delta = (\varepsilon_0, \infty)$ for some $\varepsilon_0 > 0$ if $x_\eta = \infty$.

When $x_\eta < \infty$, since η is not degenerate at zero, it holds that $x_\eta > 0$. It follows from the definition of x_η that for any $\delta \in (0, x_\eta)$, $\mathbf{P}(\eta \in (\delta, x_\eta]) > 0$. By Definition 1, let $\varepsilon_0 = x_\eta - \varepsilon$ for some small $\varepsilon > 0$ and $\Delta = (\varepsilon_0, x_\eta]$. Then $\varepsilon_0 \in (0, x_\eta)$, $\inf_{y\in\Delta} h(y) \in (0,\infty)$, and $\mathbf{P}(\eta \in \Delta) > 0$.

When $x_\eta = \infty$, it follows from the definition of x_η that for any $\delta > 0$, $\mathbf{P}(\eta \in (\delta,\infty)) > 0$. By Definition 1, let $\varepsilon_0 = D$ for some large $D > 0$ and $\Delta = (\varepsilon_0, \infty)$. Then $\inf_{y\in\Delta} h(y) \in (0,\infty)$ and $\mathbf{P}(\eta \in \Delta) > 0$.

Therefore, by the above result and $V_1 \in \mathscr{D}$, it holds that

$$\begin{aligned}
\mathbf{E}\left[h(\eta)\overline{V}_{1*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right] &\geqslant \inf_{y\in\Delta} h(y) \int_{\Delta} \overline{V}_{1*}\left(\frac{1}{y}\right)\mathbf{P}(\eta \in \mathrm{d}y) \\
&\geqslant \inf_{y\in\Delta} h(y)\overline{V}_{1*}\left(\frac{1}{\varepsilon_0}\right)\mathbf{P}(\eta \in \Delta) \\
&> 0.
\end{aligned}$$

Since $V_1 \in \mathscr{D}$, by Lemma 1, for $p > J^+_{V_1}$, there exist positive constants C_1 and D_1 such that for all $x \geqslant y \geqslant D_1$,

$$\frac{\overline{V}_1(y)}{\overline{V}_1(x)} \leqslant C_1\left(\frac{y}{x}\right)^{-p}. \tag{5}$$

Since (ξ,η) satisfies the dependence structure $\mathscr{H}$, for any $0 < \varepsilon < 1$, there exists a positive constant $D_2 > D_1 \vee 1$ such that for any $0 < y \leqslant x/D_2$,

$$(1-\varepsilon)\overline{V}_1\left(\frac{x}{y}\right)h(y) \leqslant \mathbf{P}\left(\xi > \frac{x}{y} \;\Big|\; \eta = y\right) \leqslant (1+\varepsilon)\overline{V}_1\left(\frac{x}{y}\right)h(y). \tag{6}$$

For any $x > D_2 y$, it holds that

$$\mathbf{P}(\xi\eta > x) = \left(\int_0^{x/D_2} + \int_{x/D_2}^{\infty}\right)\mathbf{P}\left(\xi > \frac{x}{y} \;\Big|\; \eta = y\right)\mathbf{P}(\eta \in \mathrm{d}y) = I_1(x) + I_2(x). \tag{7}$$

We will prove the lower bound of $\mathbf{P}(\xi\eta > x)$. By Fatou's lemma and (6),

$$\begin{aligned}
\liminf_{x\to\infty} \frac{\mathbf{P}(\xi\eta > x)}{\overline{V}_1(x)} &\geqslant (1-\varepsilon)\liminf_{x\to\infty} \int_0^{x/D_2} \frac{\overline{V}_1(x/y)}{\overline{V}_1(x)} h(y)\,\mathbf{P}(\eta \in \mathrm{d}y) \\
&\geqslant (1-\varepsilon)\int_0^{\infty} \liminf_{x\to\infty} \frac{\overline{V}_1(x/y)}{\overline{V}_1(x)} h(y)\,\mathbf{P}(\eta \in \mathrm{d}y) \\
&= (1-\varepsilon)\mathbf{E}\left[h(\eta)\overline{V}_{1*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right].
\end{aligned}$$

By the arbitrariness of ε, it holds that

$$\liminf_{x\to\infty}\frac{\mathbf{P}(\xi\eta>x)}{\overline{V}_1(x)}\geqslant\mathbf{E}\left[h(\eta)\overline{V}_{1*}\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right].$$

(ii) Now we prove the upper bound of $\mathbf{P}(\xi\eta>x)$. Since $\overline{V}_2(x)\underset{x\to\infty}{=}o(\overline{V}_1(x))$, by $V_1\in\mathscr{D}$, it holds that

$$\limsup_{x\to\infty}\frac{I_2(x)}{\overline{V}_1(x)}\leqslant\limsup_{x\to\infty}\frac{\overline{V}_2(x/D_2)}{\overline{V}_1(x/D_2)}\limsup_{x\to\infty}\frac{\overline{V}_1(x/D_2)}{\overline{V}_1(x)}=0. \tag{8}$$

In the following, we deal with $I_1(x)$. When $0<y\leqslant 1$, it holds for any $x>0$ that

$$\frac{\overline{V}_1(x/y)}{\overline{V}_1(x)}\leqslant 1, \tag{9}$$

and when $1<y\leqslant x/D_2$, by (5), it holds for any $x>D_2$ that

$$\frac{\overline{V}_1(x/y)}{\overline{V}_1(x)}\leqslant C_1y^p. \tag{10}$$

By (9) and (10), for any $y>0$, $\overline{V}_1^*(1/y)\leqslant 1\vee(C_1y^p)$. Hence,

$$\begin{aligned}\mathbf{E}\left[h(\eta)\overline{V}_1^*\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right]&\leqslant(1\vee C_1)\mathbf{E}\big[h(\eta)\big(1\vee\eta^p\big)\mathbf{1}_{\{\eta>0\}}\big]\\&\leqslant 2(1\vee C_1)\big(1\vee\mathbf{E}\big[h(\eta)\eta^p\mathbf{1}_{\{\eta>0\}}\big]\big)<\infty.\end{aligned}$$

Since $\mathbf{E}[h(\eta)\eta^p]<\infty$, by (6), (9), (10), and the dominated convergence theorem, it holds that

$$\begin{aligned}\limsup_{x\to\infty}\frac{I_1(x)}{\overline{V}_1(x)}&\leqslant(1+\varepsilon)\int_0^\infty\limsup_{x\to\infty}\frac{\overline{V}_1(x/y)}{\overline{V}_1(x)}h(y)\mathbf{P}(\eta\in\mathrm{d}y)\\&=(1+\varepsilon)\mathbf{E}\left[h(\eta)\overline{V}_1^*\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right].\end{aligned} \tag{11}$$

By the arbitrariness of ε, it follows from (7), (8), and (11) that

$$\limsup_{x\to\infty}\frac{\mathbf{P}(\xi\eta>x)}{\overline{V}_1(x)}\leqslant\mathbf{E}\left[h(\eta)\overline{V}_1^*\left(\frac{1}{\eta}\right)\mathbf{1}_{\{\eta>0\}}\right].$$

By $V_1\in\mathscr{D}$ and the results of Lemma 2 (i) and (ii), it is easy to show that the distribution function of $\xi\eta$ belongs to the class $\mathscr{D}$. This completes the proof of Lemma 2. $\square$

Lemma 3. *Let ξ be a real-valued random variable with a distribution function V, and let η be a nonnegative nondegenerate at zero random variable. Assume that the random pair (ξ,η) satisfies the dependence structure $\mathscr{H}$ with a nonnegative function $h(\cdot)$ and there exist positive constants D and C such that for all $x > D$ and $y \in R(\eta)$,*

$$\mathbf{P}(\xi < -x|\eta = y) \leqslant C\mathbf{P}(\xi > x \mid \eta = y). \tag{12}$$

If $V \in \mathscr{D}$ and $\mathbf{E}[\eta^p] < \infty$ for some $p > J_V^+$, then for any $s > 0$,

$$\mathbf{P}(\xi\eta < -xs) \underset{x\to\infty}{=} O\big(\mathbf{P}\big(\xi\eta > x\big)\big).$$

Proof. Let us take a constant q that satisfies the condition $J_F^+ p^{-1} < q < 1$.

$$\begin{aligned}\mathbf{P}(\xi\eta < -xs) &= \int_0^{x^q} \mathbf{P}\biggl(\xi < -\frac{xs}{y} \biggm| \eta = y\biggr)\mathbf{P}(\eta \in \mathrm{d}y) + \mathbf{P}(\xi\eta < -xs,\, \eta > x^q) \\ &=: I_1(x,s) + I_2(x,s).\end{aligned} \tag{13}$$

By Markov's inequality, Lemma 2(i), $\mathbf{E}[\eta^p] < \infty$, and Lemma 1,

$$\begin{aligned}\limsup_{x\to\infty} \frac{I_2(x,s)}{\mathbf{P}(\xi\eta > x)} &\leqslant \limsup_{x\to\infty} \frac{x^{-pq}}{\overline{V}(x)} \frac{\overline{V}(x)}{\mathbf{P}(\xi\eta > x)}\mathbf{E}\bigl[\eta^p\bigr] \\ &\leqslant \limsup_{x\to\infty} \frac{x^{-pq}}{\overline{V}(x)} \biggl[\mathbf{E}\biggl[h(\eta)\overline{V}_{1*}\biggl(\frac{1}{\eta}\biggr)\mathbf{1}_{\{\eta>0\}}\biggr]\biggr]^{-1}\mathbf{E}\bigl[\eta^p\bigr] \\ &= 0.\end{aligned} \tag{14}$$

By (12) and (3), it holds for any $s > 0$ and for sufficiently large x that

$$\begin{aligned}I_1(x,s) &\leqslant C\int_0^{x^q} \mathbf{P}\biggl(\xi > \frac{xs}{y} \biggm| \eta = y)\,\mathbf{P}(\eta \in \mathrm{d}y) \sim C\int_0^{x^q} \overline{V}\biggl(\frac{xs}{y}\biggr)h(y)\,\mathbf{P}(\eta \in \mathrm{d}y) \\ &\lesssim C \sup_{0<y\leqslant x^q} \frac{\overline{V}(xs/y)}{\overline{V}(x/y)} \int_0^{x^q} \mathbf{P}\biggl(\xi > \frac{x}{y} \biggm| \eta = y\biggr)\mathbf{P}(\eta \in \mathrm{d}y) \\ &\leqslant C \sup_{0<y\leqslant x^q} \frac{\overline{V}(xs/y)}{\overline{V}(x/y)}\,\mathbf{P}(\xi\eta > x),\end{aligned} \tag{15}$$

where the constant C might vary from line to line. Since $V \in \mathscr{D}$, by (15), it holds for any $s > 0$ that

$$\limsup_{x\to\infty} \frac{I_1(x,s)}{\mathbf{P}(\xi\eta > x)} \leqslant C \limsup_{x\to\infty} \frac{\overline{V}(xs)}{\overline{V}(x)} < \infty,$$

which, combined with (13) and (14), yields that

$$\mathbf{P}(\xi\eta < -xs) \underset{x\to\infty}{=} O\bigl(\mathbf{P}(\xi\eta > x)\bigr).$$

This completes the proof of Lemma 3. $\square$

The following result is (2.1) of [17].

Lemma 4. *Let V be a distribution function on $(-\infty, \infty)$. If $V \in \mathscr{L}$, then there is a function $l(\cdot) : [0, \infty) \to [0, \infty)$ such that the following statements hold simultaneously as $x \to \infty$:*

$$l(x) \to \infty, \quad l(x) = o(x), \quad \textit{and} \quad \overline{V}\big(x \pm l(x)\big) \sim \overline{V}(x).$$

The following lemma is Theorem 3.1 of [17].

Lemma 5. *Let ξ_i, $i = 1, 2, \dots, n$, be n real-valued random variables with distribution functions V_i, $i = 1, 2, \dots, n$, respectively. If $V_i \in \mathscr{L} \cap \mathscr{D}$ for all $i = 1, 2, \dots, n$ and*

$$\lim_{\min\{x_i, x_j\} \to \infty} \mathbf{P}\big(|\xi_i| > x_i \bigm| \xi_j > x_j\big) = 0$$

holds for all $1 \leqslant i \neq j \leqslant n$, then

$$\mathbf{P}\left(\sum_{i=1}^{n} \xi_i > x\right) \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \overline{V}_i(x).$$

The following lemma presents a result of randomly weighted sums with random variables satisfying the dependence structure $\mathscr{H}$.

Lemma 6. *Let ξ_i, $i = 1, 2, \dots, n$, be n real-valued random variables with distribution functions V_i, $i = 1, 2, \dots, n$, respectively, and let η_i, $i = 1, 2, \dots, n$, be another n nonnegative nondegenerate at zero random variables with distribution functions W_i, $i = 1, 2, \dots, n$, respectively. Assume that (ξ_i, η_i), $i = 1, 2, \dots, n$, are independent random vectors and for each $i = 1, 2, \dots, n$, (ξ_i, η_i) satisfies the dependence structure $\mathscr{H}$ with a nonnegative function $h_i(\cdot)$. If for any $i = 1, 2, \dots, n$, $V_i \in \mathscr{L} \cap \mathscr{D}$, $\overline{W}_i(x) \underset{x\to\infty}{=} o(\overline{V}_i(x))$, and $\mathbf{E}[h_i(\eta_i)\eta_i^p] < \infty$ for some $p > \vee_{i=1}^{n} J_{V_i}^{+}$, then*

$$\mathbf{P}\left(\sum_{i=1}^{n} \xi_i\eta_i > x\right) \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \mathbf{P}(\xi_i\eta_i > x).$$

Remark 2. When there exists a positive constant b such that $\mathbf{P}(0 \leqslant \eta_i \leqslant b) = 1$ and $V_i(x) = V(x), x \in (-\infty, \infty)$, Theorem 1.2 of [31] yields the above asymptotic result for $V \in \mathscr{S}$.

Proof of Lemma 6. For any $i = 1, 2, \dots, n$, since $V_i \in \mathscr{D}$ and $\overline{W}_i(x) \underset{x\to\infty}{=} o(\overline{V}_i(x))$, it holds for all $c > 0$ that

$$\overline{W}_i(x) \underset{x\to\infty}{=} o\big(\overline{V}_i(x)\big) \underset{x\to\infty}{=} o\big(\overline{V}_i(cx)\big),$$

which, combined with $\mathbf{E}[h_i(\eta_i)\eta_i^p] < \infty$ for some $p > \vee_{i=1}^{n} J_{V_i}^{+}$, means that the conditions of Lemma 2 are satisfied. Hence, by Lemma 2(ii) and (iii), we get that for all

$i = 1, 2, \dots, n$, the distribution function of $\xi_i\eta_i$ belongs to the class $\mathscr{L} \cap \mathscr{D}$. Since $\{\xi_i\eta_i,\ i = 1, 2, \dots, n\}$ are independent, by Lemma 5 for the independent case, it holds that

$$\mathbf{P}\left(\sum_{i=1}^{n} \xi_i\eta_i > x\right) \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \mathbf{P}(\xi_i\eta_i > x). \qquad \square$$

Lemma 7. *Under the conditions of Theorem* 1*, for any* $u > 0$*, it holds that*

$$\mathbf{P}\left(X_k\theta_k > ux,\ \sum_{i=1}^{n} X_i\theta_i > x\right) \underset{x\to\infty}{\sim} \mathbf{P}\big(X_k\theta_k > (u \vee 1)x\big).$$

Proof. Without loss of generality, we consider only $k = 1$. Let $Z_i = X_i\theta_i, i = 1, 2, \dots, n$. We will prove

$$\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) \underset{x\to\infty}{\sim} \mathbf{P}\big(Z_1 > (u \vee 1)x\big).$$

When $0 < u \leqslant 1$, we want to prove

$$\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) \underset{x\to\infty}{\sim} \mathbf{P}(Z_1 > x). \tag{16}$$

For any $x > 0$, it holds that

$$\begin{aligned}
\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) &\leqslant \mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i^+ > x\right) \\
&= \mathbf{P}\left(\sum_{i=1}^{n} Z_i^+ > x\right) - \mathbf{P}\left(Z_1 \leqslant ux,\ \sum_{i=1}^{n} Z_i^+ > x\right) \\
&=: J_1(x) - J_2(x).
\end{aligned} \tag{17}$$

It follows from Lemma 6 that

$$J_1(x) \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \mathbf{P}(Z_i > x). \tag{18}$$

For $J_2(x)$, since Z_i, $i = 1, 2, \dots, n$, are independent, it holds for sufficiently large x that

$$\begin{aligned}
J_2(x) &\geqslant \mathbf{P}\left(Z_1 \leqslant ux,\ \bigcup_{i=1}^{n} \{Z_i^+ > x\}\right) \\
&\geqslant \sum_{i=2}^{n} \mathbf{P}(Z_i > x) - \mathbf{P}(Z_1 > ux) \sum_{i=2}^{n} \mathbf{P}(Z_i > x) - \sum_{i=2}^{n} \mathbf{P}(Z_i > x) \sum_{j=2}^{n} \mathbf{P}(Z_j > x) \\
&\underset{x\to\infty}{\sim} \sum_{i=2}^{n} \mathbf{P}(Z_i > x).
\end{aligned} \tag{19}$$

By (18) and (19), for any $\varepsilon > 0$, there exists a constant $x_0 > 0$ such that for all $x \geqslant x_0$,

$$J_1(x) \leqslant (1+\varepsilon)\sum_{i=1}^{n}\mathbf{P}(Z_i > x) \quad \text{and} \quad J_2(x) \geqslant (1-\varepsilon)\sum_{i=2}^{n}\mathbf{P}(Z_i > x),$$

which, combined with (17), yields that for any $x \geqslant x_0$,

$$\mathbf{P}(Z_1 > ux, \sum_{i=1}^{n} Z_i > x) \leqslant (1+\varepsilon)\mathbf{P}(Z_1 > x) + 2\varepsilon\sum_{i=2}^{n}\mathbf{P}(Z_i > x). \tag{20}$$

Since for any $i = 2, \dots, n$, $\overline{F}_i(x) \underset{x\to\infty}{=} O(\overline{F}_1(x))$, by Lemma 2(i) and (ii), it holds that

$$\sum_{i=2}^{n}\mathbf{P}(Z_i > x) \underset{x\to\infty}{=} O\big(\mathbf{P}(Z_1 > x)\big).$$

Thus, by (20) and the arbitrariness of ε, it holds that

$$\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) \underset{x\to\infty}{\lesssim} \mathbf{P}(Z_1 > x). \tag{21}$$

In the following, we will prove

$$\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) \underset{x\to\infty}{\gtrsim} \mathbf{P}(Z_1 > x). \tag{22}$$

Since $F_1 \in \mathscr{L}$ and $\overline{G}_1(x) \underset{x\to\infty}{=} o(\overline{F}_1(x))$, by Lemma 2(iii), it can be obtained that the distribution function of Z_1 belongs to the class $\mathscr{L}$. Thus, by Lemma 4, there exists a nonnegative function $l(\cdot)$ such that $l(x) \to \infty$ as $x \to \infty$ and

$$\mathbf{P}\big(Z_1 > x + l(x)\big) \underset{x\to\infty}{\sim} \mathbf{P}(Z_1 > x). \tag{23}$$

Therefore, for any $x > 0$,

$$\begin{aligned}\mathbf{P}\left(Z_1 > ux,\ \sum_{i=1}^{n} Z_i > x\right) &\geqslant \mathbf{P}\left(Z_1 > x + l(x),\ \sum_{i=1}^{n} Z_i > x\right)\\ &= \mathbf{P}\big(Z_1 > x + l(x)\big)\\ &\quad - \mathbf{P}\left(Z_1 > x + l(x),\ \sum_{i=1}^{n} Z_i \leqslant x\right).\end{aligned} \tag{24}$$

Since Z_i, $i = 1, 2, \dots, n$, are independent, for sufficiently large x, it holds that

$$\begin{aligned}\mathbf{P}\left(Z_1 > x + l(x),\ \sum_{i=1}^{n} Z_i \leqslant x\right) &\leqslant \mathbf{P}\left(Z_1 > x + l(x),\ \sum_{i=2}^{n} Z_i \leqslant -l(x)\right)\\ &\leqslant \mathbf{P}\big(Z_1 > x + l(x)\big)\sum_{i=1}^{n}\mathbf{P}\left(Z_i \leqslant -\frac{l(x)}{n-1}\right)\\ &\underset{x\to\infty}{=} o\big(\mathbf{P}\big(Z_1 > x + l(x)\big)\big).\end{aligned} \tag{25}$$

By (23)–(25), we get that (22) holds. This completes the proof of (16) by (21) and (22).

In the following, we will consider the case of $u > 1$ and prove

$$\mathbf{P}\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \underset{x\to\infty}{\sim} \mathbf{P}(Z_1 > ux). \tag{26}$$

It is obvious that for any $x > 0$,

$$\mathbf{P}\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) \leqslant \mathbf{P}(Z_1 > ux). \tag{27}$$

Since Z_i, $i = 1, 2, \dots, n$, are independent, it holds for sufficiently large x that

$$\begin{aligned}\mathbf{P}\left(Z_1 > ux, \sum_{i=1}^{n} Z_i > x\right) &\geqslant \mathbf{P}(Z_1 > ux) - \mathbf{P}\left(Z_1 > ux, \sum_{i=2}^{n} Z_i \leqslant (1-u)x\right) \\ &\geqslant \mathbf{P}(Z_1 > ux)\left(1 - \sum_{i=2}^{n} \mathbf{P}\left(Z_i \leqslant \frac{(1-u)x}{n-1}\right)\right) \\ &\underset{x\to\infty}{\sim} \mathbf{P}(Z_1 > ux),\end{aligned}$$

which, combined with (27), yields that (26) holds. Therefore, the proof of Lemma 7 is complete. □

Lemma 8. *Under the conditions of Theorem* 1*, it holds that*

$$\lim_{x\to\infty} \frac{\mathbf{E}[X_k\theta_k\mathbf{1}_{\{S_n>x\}}]}{\mathbf{E}[X_k\theta_k\mathbf{1}_{\{X_k\theta_k>x\}}]} = \lim_{x\to\infty} \frac{\mathbf{E}[X_k^+\theta_k\mathbf{1}_{\{S_n>x\}}]}{\mathbf{E}[X_k\theta_k\mathbf{1}_{\{X_k\theta_k>x\}}]} = 1.$$

Proof. For $Z_i = X_i\theta_i$, $i = 1, 2, \dots, n$, by the conditions of Theorem 1, we know that the conditions of Lemma 2 are satisfied. Hence, for all $i = 1, 2, \dots, n$, the distribution functions of Z_i belong to the class $\mathscr{L} \cap \mathscr{D}$, and $\mathbf{P}(Z_i > x) \underset{x\to\infty}{\asymp} \overline{F}_i(x)$. Since Z_i, $i = 1, 2, \dots, n$, are independent, by Lemmas 1–4, this lemma can be proved along the same lines as in the proof of Lemma A.4 of [8]. We omit the details. □

Lemma 9. *Under the conditions of Theorem* 1*, for the* ES*capital allocation, it holds that*

$$\lim_{q\uparrow 1} \frac{A_k(q)}{\mathbf{E}[Z_k | S_n > H_n^{\leftarrow}(q)]} = 1.$$

Proof. By Lemma 2 and the conditions of Theorem 1, we get that for all $i = 1, 2, \dots, n$, the distribution function of $X_i\theta_i$ belongs to the class of $\mathscr{L} \cap \mathscr{D}$. Therefore, by Lemma 6, it holds that

$$\mathbf{P}(S_n > x) \underset{x\to\infty}{\sim} \sum_{i=1}^{n} \mathbf{P}(X_i\theta_i > x),$$

which, combined with the distribution function of $X_i\theta_i$, $i = 1, 2, \dots, n$, belongs to the class of $\mathscr{L} \cap \mathscr{D}$, yields that $H_n \in \mathscr{L} \cap \mathscr{D}$. Hence, by (A.2) of [8], it holds that

$$\mathbf{P}\big(S_n > H_n^{\leftarrow}(q)\big) \underset{q\uparrow 1}{\sim} \mathbf{P}\big(S_n \geqslant H_n^{\leftarrow}(q)\big) \underset{q\uparrow 1}{\sim} 1 - q. \tag{28}$$

So, by Lemma 8 and along the steps of the proof of Lemma A.5 of [8], this lemma can be proved. We omit the details. □

5.2 Proof of Theorem 1

We will use the same lines as in the proof of Theorem 3.1 of [8]. For any $0 < q < 1$, by (1),

$$\begin{aligned} \mathrm{SES}_k(q) &= \frac{\int_{A_k(q)}^{\infty} \mathbf{P}(Z_k > z, S_n > A(q))\mathrm{d}z}{\mathbf{P}(S_n > A(q))} \\ &= \frac{A(q)}{\mathbf{P}(S_n > A(q))}\left(\int\limits_{A_k(q)/A(q)}^{1} + \int\limits_{1}^{\infty}\right)\mathbf{P}\big(Z_k > uA(q),\, S_n > A(q)\big)\,\mathrm{d}u \\ &=: \frac{A(q)}{\mathbf{P}(S_n > A(q))}\big(Q_1(q) + Q_2(q)\big). \end{aligned} \tag{29}$$

We first deal with $Q_1(q)$. Since $\liminf_{q\uparrow 1} A_k(q)/A(q) =: a_k > 0$, for any $0 < \delta < 1$, there exists a positive constant $q_0 \in (0, 1)$ such that for any $q \in (q_0, 1)$,

$$\frac{A_k(q)}{A(q)} \geqslant a_k\delta. \tag{30}$$

We will show that

$$\mathbf{P}\big(Z_k > uA(q),\, S_n > A(q)\big) \underset{q\uparrow 1}{\sim} \mathbf{P}\big(Z_k > A(q)\big) \tag{31}$$

holds uniformly for $u \in [a_k\delta, 1]$. It follows from Lemma 7 that

$$\mathbf{P}\big(Z_k > uA(q),\, S_n > A(q)\big) \underset{q\uparrow 1}{\gtrsim} \mathbf{P}\big(Z_k > A(q)\big)$$

and

$$\begin{aligned} \mathbf{P}\big(Z_k > uA(q),\, S_n > A(q)\big) &\leqslant \mathbf{P}\big(Z_k > a_k\delta A(q),\, S_n > A(q)\big) \\ &\underset{q\uparrow 1}{\sim} \mathbf{P}\big(Z_k > A(q)\big). \end{aligned}$$

Therefore, by (30), (31) holds uniformly for $u \in [A_k(q)/A(q), 1]$, which means that

$$Q_1(q) \underset{q\uparrow 1}{\sim} \int\limits_{A_k(q)/A(q)}^{1} \mathbf{P}\big(Z_k > A(q)\big)\,\mathrm{d}u = \left(1 - \frac{A_k(q)}{A(q)}\right)\mathbf{P}\big(Z_k > A(q)\big). \tag{32}$$

For $Q_2(q)$, using the way of the proof of (A.30) in [8], by Lemmas 2 and 7, it can be obtained that

$$Q_2(q) \underset{q\uparrow 1}{\sim} \int_1^\infty \mathbf{P}\bigl(Z_k > uA(q)\bigr)\,\mathrm{d}u. \tag{33}$$

By Lemma 6, it holds that

$$\mathbf{P}\bigl(S_n > A(q)\bigr) \underset{q\uparrow 1}{\sim} \sum_{i=1}^n \mathbf{P}\bigl(Z_i > A(q)\bigr). \tag{34}$$

By (29) and (32)–(34), it holds that

$$\begin{aligned}
\mathrm{SES}_k(q) &\underset{q\uparrow 1}{\sim} \frac{A(q)}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))} \\
&\quad \times \left(\left(1 - \frac{A_k(q)}{A(q)}\right)\mathbf{P}\bigl(Z_k > A(q)\bigr) + \int_1^\infty \mathbf{P}\bigl(Z_k > uA(q)\bigr)\,\mathrm{d}u\right) \\
&= \frac{\mathbf{P}(Z_k > A(q))\sum_{i=1,i\neq k}^n A_i(q) + \mathbf{E}[(Z_k - A(q))^+]}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))} \\
&= \frac{\mathbf{E}[Z_k \mathbf{1}_{\{Z_k > A(q)\}}] - A_k(q)\mathbf{P}(Z_k > A(q))}{\sum_{i=1}^n \mathbf{P}(Z_i > A(q))}.
\end{aligned}$$

For the estimation of $\mathrm{MES}_k(q)$, by (2) and Lemmas 8 and 6, it holds that

$$\begin{aligned}
\mathrm{MES}_k(q) = \mathbf{E}\bigl[Z_k \bigm| S_n > A(q)\bigr] &= \frac{\mathbf{E}[Z_k \mathbf{1}_{\{S_n > A(q)\}}]}{\mathbf{P}(S_n > A(q))} \\
&\underset{q\uparrow 1}{\sim} \frac{\mathbf{E}[Z_k \mathbf{1}_{\{Z_k > A(q)\}}]}{\sum_{k=1}^n \mathbf{P}(Z_k > A(q))}.
\end{aligned}$$

This ends the proof of Theorem 1.

5.3 Proof of Corollary 1

By Lemma 6, Remark 1(1), and $\overline{F}_i(x) \underset{x\to\infty}{\sim} b_i\overline{F}(x)$, $i = 1, 2, \dots, n$, it holds that

$$\begin{aligned}
\overline{H}_n(x) = \mathbf{P}(S_n > x) &\underset{x\to\infty}{\sim} \sum_{i=1}^n \mathbf{P}(X_i\theta_i > x) \\
&\underset{x\to\infty}{\sim} \sum_{i=1}^n b_i \mathbf{E}\bigl[h_i(\theta_i)\theta_i^\alpha \mathbf{1}_{\{\theta_i > 0\}}\bigr]\overline{F}(x) =: C_n\overline{F}(x).
\end{aligned} \tag{35}$$

By (35) and $F \in \mathscr{R}_{-\alpha}$, it holds that $F^{\leftarrow} \in \mathscr{R}_{-1/\alpha}$ and $H_n^{\leftarrow} \in \mathscr{R}_{-1/\alpha}$. By (35), Proposition 0.8(v) of [23], and $F^{\leftarrow} \in \mathscr{R}_{-1/\alpha} \subset \mathscr{L}$, we can get that

$$H_n^{\leftarrow}(q) \underset{q\uparrow 1}{\sim} F^{\leftarrow}\bigl(1 - C_n^{-1}(1-q)\bigr) \underset{q\uparrow 1}{\sim} F^{\leftarrow}\bigl(C_n^{-1}(1-q)\bigr) \underset{q\uparrow 1}{\sim} C_n^{1/\alpha} F^{\leftarrow}(q).$$

Since $F \in \mathscr{R}_{-\alpha}$, by (35), we get that $H_n \in \mathscr{R}_{-\alpha} \subset \mathscr{L}$. By Lemma 9, (28), Lemma 8, Remark 1(1), $F \in \mathscr{R}_{-\alpha}$, and (35), it holds for each $k = 1, 2, \ldots, n$ that

$$\begin{aligned} A_k(q) &\underset{q\uparrow 1}{\sim} \mathbf{E}\big[Z_k | S_n > H_n^{\leftarrow}(q)\big] \underset{q\uparrow 1}{\sim} \frac{1}{1-q}\mathbf{E}\big[Z_k \mathbf{1}_{\{Z_k > H_n^{\leftarrow}(q)\}}\big] \\ &\underset{q\uparrow 1}{\sim} \frac{b_k \mathbf{E}[h_k(\theta_k)\theta_k^{\alpha}\mathbf{1}_{\{\theta_k>0\}}]}{1-q} \int_0^{\infty} \overline{F}\big(z \vee H_n^{\leftarrow}(q)\big)\,\mathrm{d}z \\ &\underset{q\uparrow 1}{\sim} \frac{b_k \mathbf{E}[h_k(\theta_k)\theta_k^{\alpha}\mathbf{1}_{\{\theta_k>0\}}]}{1-q} H_n^{\leftarrow}(q)\overline{F}\big(H_n^{\leftarrow}(q)\big) \int_0^{\infty} (u \vee 1)^{-\alpha}\,\mathrm{d}u \\ &\underset{q\uparrow 1}{\sim} b_k \mathbf{E}\big[h_k(\theta_k)\theta_k^{\alpha}\mathbf{1}_{\{\theta_k>0\}}\big] C_n^{1/\alpha-1} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q). \end{aligned} \tag{36}$$

By (36), it holds that

$$\begin{aligned} A(q) &\underset{q\uparrow 1}{\sim} \sum_{i=1}^{n} b_i \mathbf{E}\big[h_i(\theta_i)\theta_i^{\alpha}\mathbf{1}_{\{\theta_i>0\}}\big] C_n^{1/\alpha-1} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q) \\ &= C_n^{1/\alpha} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q). \end{aligned} \tag{37}$$

Then for any $\varepsilon \in (0,1)$, there exists a positive constant $q_0 \in (0,1)$ such that for any $q \in (q_0, 1)$,

$$(1-\varepsilon) C_n^{1/\alpha} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q) \leqslant A(q) \leqslant (1+\varepsilon) C_n^{1/\alpha} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q). \tag{38}$$

Since $F \in \mathscr{R}_{-\alpha} \subset \mathscr{L}$, by (38), it holds that

$$\begin{aligned} \overline{F}\big(A(q)\big) &\leqslant \overline{F}\bigg((1-\varepsilon) C_n^{1/\alpha} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q)\bigg) \\ &\underset{q\uparrow 1}{\sim} (1-\varepsilon)^{-\alpha} C_n^{-1} \bigg(\frac{\alpha}{\alpha-1}\bigg)^{-\alpha} (1-q) \end{aligned}$$

and

$$\begin{aligned} \overline{F}\big(A(q)\big) &\geqslant \overline{F}\bigg((1+\varepsilon) C_n^{1/\alpha} \frac{\alpha}{\alpha-1} F^{\leftarrow}(q)\bigg) \\ &\underset{q\uparrow 1}{\sim} (1+\varepsilon)^{-\alpha} C_n^{-1} \bigg(\frac{\alpha}{\alpha-1}\bigg)^{-\alpha} (1-q), \end{aligned}$$

which, combined with the arbitrariness of ε, yields that

$$\overline{F}\big(A(q)\big) \underset{q\uparrow 1}{\sim} C_n^{-1} \bigg(\frac{\alpha}{\alpha-1}\bigg)^{-\alpha} (1-q). \tag{39}$$

By (35) and (39), we get that

$$\mathbf{P}\big(S_n > A(q)\big) \underset{q\uparrow 1}{\sim} C_n \overline{F}\big(A(q)\big) \underset{q\uparrow 1}{\sim} \bigg(\frac{\alpha}{\alpha-1}\bigg)^{-\alpha} (1-q). \tag{40}$$

Since $F \in \mathscr{R}_{-\alpha}$ and $\overline{F}_k(x) \underset{x\to\infty}{\sim} b_k\overline{F}(x)$, it follows from Remark 1(1) and (39) that

$$\begin{aligned}\mathbf{P}\big(Z_k > A(q)\big) &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]\overline{F}\big(A(q)\big)\\ &\underset{q\uparrow 1}{\sim} \frac{b_k\mathbf{E}[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}]}{C_n}\left(\frac{\alpha}{\alpha-1}\right)^{-\alpha}(1-q).\end{aligned} \tag{41}$$

By Remark 1(1), $\overline{F}_k(x) \underset{x\to\infty}{\sim} b_k\overline{F}(x)$, $F \in \mathscr{R}_{-\alpha}$, (39), and (37), it holds that

$$\begin{aligned}\mathbf{E}\big[(Z_k - A(q))^+\big] &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]A(q)\int\limits_0^\infty \overline{F}\big((1+u)A(q)\big)\,\mathrm{d}u\\ &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]C_n^{1/\alpha-1}\frac{\alpha^{1-\alpha}}{(\alpha-1)^{2-\alpha}}(1-q)F^{\leftarrow}(q).\end{aligned}$$

Hence, it follows from Theorem 1, Lemma 6, (40), (41), and (36) that

$$\begin{aligned}\mathrm{SES}_k(q) &\underset{q\uparrow 1}{\sim} \frac{\mathbf{P}(Z_k > A(q))\sum_{i=1,i\neq k}^n A_i(q) + \mathbf{E}[(Z_k - A(q))^+]}{\mathbf{P}(S_n > A(q))}\\ &\underset{q\uparrow 1}{\sim} C_n^{1/\alpha-1}\frac{\alpha}{\alpha-1}b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]\\ &\quad\times\left(C_n^{-1}\sum_{i=1,\,i\neq k}^n b_i\mathbf{E}\big[h_i(\theta_i)\theta_i^\alpha \mathbf{1}_{\{\theta_i>0\}}\big] + \frac{1}{\alpha-1}\right)F^{\leftarrow}(q).\end{aligned}$$

By Remark 1(i), $\overline{F}_k(x) \underset{x\to\infty}{\sim} b_k\overline{F}(x)$, $F \in \mathscr{R}_{-\alpha}$, (37), and (39), it holds that

$$\begin{aligned}\mathbf{E}[Z_k\mathbf{1}_{\{Z_k>A(q)\}}] &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]\int\limits_0^\infty \overline{F}\big(z\vee A(q)\big)\,\mathrm{d}z\\ &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]A(q)\overline{F}\big(A(q)\big)\frac{\alpha}{\alpha-1}\\ &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]C_n^{1/\alpha-1}\left(\frac{\alpha}{\alpha-1}\right)^{2-\alpha}(1-q)F^{\leftarrow}(q),\end{aligned}$$

which, combined with Theorem 1, Lemma 6, and (40), yields that

$$\begin{aligned}\mathrm{MES}_k(q) &\underset{q\uparrow 1}{\sim} \frac{\mathbf{E}[Z_k\mathbf{1}_{\{Z_k>A(q)\}}]}{\mathbf{P}(S_n > A(q))}\\ &\underset{q\uparrow 1}{\sim} b_k\mathbf{E}\big[h_k(\theta_k)\theta_k^\alpha \mathbf{1}_{\{\theta_k>0\}}\big]C_n^{1/\alpha-1}\left(\frac{\alpha}{\alpha-1}\right)^{2}F^{\leftarrow}(q).\end{aligned}$$

This completes the proof of Corollary 1.

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