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Approximate controllability of Caputo fractional Volterra–Fredholm integral equations under nonlocal conditions and impulsive effects

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Abstract. This paper investigates the approximate controllability of a system of fractional control differential equations. The system involves Caputo fractional derivatives, Volterra–Fredholm integral equations, and impulsive effects. The analysis is carried out in the framework of Banach spaces under nonlocal conditions of order $r \in (1, 2)$. The main objective is to establish sufficient conditions for the approximate controllability of the proposed control problem. Assuming that the associated linear system is approximately controllable, the analysis relies on tools from fractional calculus, Krasnoselskii's fixed point theorem, and the theory of resolvent operators. The obtained results extend and improve several existing results in the related literature. Finally, two illustrative examples are presented to demonstrate the applicability of the theoretical findings.

Keywords: fractional derivative, fixed point technique, approximate controllability, nonlocal condition, impulsive term.

1 Introduction

The current study examines the approximate controllability of the following nonlinear fractional control system:

$$
\begin{aligned}
{}^{c}D^{r}u(\zeta) = -Bu(\zeta) + \hbar\left(\zeta, u(\zeta), \int_0^{\zeta} \varpi\big(\zeta, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^{\eta} \sigma\big(\zeta, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)& \\
+ E\varphi(\zeta), \quad \zeta \in U = [0, \eta],& \qquad (1_1)
\end{aligned}
$$

$$\begin{aligned} &u(0) + p(u) = u_0, \qquad u'(0) + q(u) = u_1, \\ &\Delta u\big|_{\zeta=\zeta_k} = J_k\big(u(\zeta_k)\big), \quad k = 1, 2, \dots, m. \end{aligned} \tag{1_2}$$

where ${}^cD^r$ represents the Caputo fractional derivative (CFD) of order $r \in (1,2)$, the infinitesimal generator of a strongly continuous r-order cosine family $\{Q_r(\zeta)\}_{r\geqslant 0}$ on a Banach space Ω is denoted by B, and the state $u(\cdot)$ takes values in Ω for $u_0, u_1 \in \Omega$. Furthermore, in a Banach space V, $E : V \to \Omega$ is a bounded linear operator (BLO), the control function $\varphi(\cdot)$ is given in $L^2(J, \mathbb{R}^+)$, and $\hbar$, ϖ, σ, p, q will be defined later. Moreover, the impulsive function $J_k : \Omega \to \Omega$ is defined for the discrete time points $0 = \zeta_1 < \zeta_2 < \cdots < \zeta_m < \zeta_{m+1} = \eta$, where $k = 1, 2, \dots, m$. The action of the impulse is denoted by $\Delta u(\zeta_k)$, which expresses the jump of the state u at time ζ_k. This jump is formally defined as $\Delta u(\zeta_k) = u(\zeta_k^+) - u(\zeta_k^-)$, where $u(\zeta_k^-)$ and $u(\zeta_k^+)$ are the left and right limits of the functions $\{u(\zeta_k)\}_{\zeta=\zeta_k}$, respectively.

Fractional differential equations (DEs) have attracted increased attention recently because they are more accurate than integer-order equations in describing a variety of events that arise in physics, engineering, economics, and other scientific fields. Fractional derivatives provide a remarkable tool for describing memory and genetic effects in a variety of materials and systems. Therefore, scientific researchers have become increasingly interested in using fractional DEs in recent decades. This increase in attention is due to notable developments in both the theory and real-world applications of these equations; see, for example, [1, 19] and the references therein.

The design and study of control systems for use in industry and research is the main emphasis of control theory, an important branch of applied mathematics. Control theory links the basic concepts of controllability to observer design, pole assignment, and structural decomposition. In order to solve particular control problems, recent research has investigated controllability in nonlinear dynamical systems in infinite-dimensional spaces. While exact controllability requires the system to be precisely steered to any desired state, approximate controllability ensures that the system can be directed to a neighborhood of the final state. The determination of a control strategy that optimizes an objective function over time is the domain of optimal control theory, which utilizes a cost function involving both the control and state variables. Further exploration of control theory is available in [8, 11, 26].

Modeling processes that exhibit abrupt changes in state at predetermined times necessitates the use of impulsive DEs. These equations are widely used to describe discontinuous dynamics in real-world systems, with applications in population dynamics, control theory, and medicine. For further details on the theory and applications of impulsive DEs, see [2, 7, 9, 14, 21].

The purpose of controllability is to identify a control function that can guide a dynamical system from an initial state to a desired final state. However, two common structural constraints complicate this: the control input frequently only impacts a portion of the system's state, and real-world monitoring typically allows for the observation of only a fraction of the whole state. As a result, determining the feasibility of regulating the entire system is a necessary step toward fixing the problem. The structural constraints in control and observation give rise to the concept of approximation controllability. The field of controllability for integer-order differential systems and fractional differential systems, specifically those of order $q \in (0,1)$, has been extensively studied and well developed in the context of Banach spaces, often using various distinct approaches, while the approximate controllability approach addresses cases where exact controllability may be unattainable. See [3, 20, 28] for more details.

As a continuation of previous work in this direction, Shukla et al. [23] has presented a good study on the approximate controllability for the following fractional problem:

$$
\begin{aligned}
&{}^{c}D^{q}u(\zeta) = -Bu(\zeta) + \hbar\bigl(\zeta, u(\zeta)\bigr) + E\varphi(\zeta), \quad \zeta \in U = [0,\eta],\\
&u(0) = u_0, \qquad u'(0) = u_1.
\end{aligned}
$$

In this context, ${}^{c}D^{q}$ represents the CFD of order $q \in (1,2)$. The operator $B : D(B) \subset \Omega \to \Omega$ is a closed linear operator defined on the Hilbert space Ω. Furthermore, B is responsible for generating a strongly continuous q-order cosine family on Ω. The findings are derived using the application of Schauder's fixed point theorem (FPT).

In [22], Shu et al. presented the approximate controllability of a fractional control system incorporating Riemann–Liouville fractional (RLF) derivatives of order $q \in (1,2)$. This was accomplished after establishing conditions ensuring the existence and uniqueness of the system's moderate solutions. Kumar [13] investigated the approximate controllability of complex fractional systems, specifically neutral integro-differential inclusions that feature noninstantaneous impulses and infinite delay. The proof relied upon using the FPT for discontinuous multivalued operators in conjunction with the q-resolvent operator. Shukla et al. [24] used the q-order cosine family theory and sequential approach to achieve approximate controllability for a fractional order system with infinite delay. Also, they used q-order cosine family theory, FPT, and stochastic analysis approaches to approximate controllability for a fractional stochastic semilinear control system with order $q \in (1,2)$ in a Banach space. The majority of the publications listed above assume conventional beginning conditions.

On the other hand, nonlocal conditions are frequently required to accurately describe specific physical processes, and they outperform the standard initial condition $u(0) = u_0$. For example, nonlocal circumstances have been successfully employed to represent diffusion phenomena [4]. Diethelm and Ford [5] provide a more in-depth assessment of uses. There has been limited research on nonlocal conditions of order $q \in (1,2)$, with the majority of studies taking place in Hilbert spaces. Li et al. [6] studied the approximate controllability of semilinear systems using composite relaxation equations in Hilbert space with nonlocal conditions, assuming the corresponding linear system is nearly controllable. Lian et al. [15] studied the approximate controllability of a fractional semilinear differential system of order $q \in (1,2)$ under nonlocal conditions in Hilbert space. Yang [27] successfully proved the approximate controllability of a Sobolev type fractional control system with nonlocal conditions in a Hilbert space. However, it is crucial to note that the study of approximate controllability for fractional nonlinear equations with nonlocal conditions in a Banach space remains in its initial stage. A notable contribution in this emerging area comes from Zoubida et al. [28], who presented significant work on the approximate controllability of a nonlinear fractional control system of order $q \in (1, 2)$ with nonlocal conditions in a Banach space.

Continuing the series of previous investigations, in this manuscript we study the effect of adding the Volterra and Fredholm integral equations with pulses to study approximate controllability for system (1). The purpose is to identify the required circumstances that make the proposed control problem roughly manageable. Assuming that the relevant linear system is roughly controllable, the discussion is based on fractional calculus, Krasnoselskii's FPT, and resolvent operator theory. The provided results improve on or outperform comparable findings in the literature on this subject. Finally, illustrative examples are given as applications to demonstrate the actual implementation of the stated results.

The rest of the paper is organized as follows: In Section 2, we present the main definitions and key auxiliary results that constitute the foundation of our forthcoming study. Section 3 is dedicated to listing the necessary and fundamental assumptions that support the achievement of the objectives of this manuscript. Section 4 addresses the existence theory of mild solutions for the fractional problem (1), where appropriate conditions ensuring existence are established. In Section 5, we investigate the approximate controllability of the control problem (1) and present the analytical techniques required to verify the obtained results. Section 6 contains two examples designed to support, clarify, and demonstrate the effectiveness of the theoretical results. Finally, in Section 7, we provide the conclusion of the manuscript.

2 Basic facts

This section highlights the main definitions and major auxiliary results that form the basis of our forthcoming study. For more information, see Podlubny [19]. In this paper, Ω denotes a Banach space with the norm $\|\cdot\|$, and $C(U, \Omega)$ is the space of continuous functions from U to Ω with the supremum norm $\|\cdot\|_C$.

Definition 1. For the function $u \in C(U, \Omega)$, the RLF integral of order $r > 0$ with the lower limit zero is given by

$$I^r u(\zeta) = \frac{1}{\Gamma(r)} \int_0^\zeta (\zeta - \kappa)^{r-1} u(\kappa)\,\mathrm{d}\kappa, \quad \zeta > 0,$$

whenever, the right-hand side is pointwise defined on $[0, \infty)$.

Definition 2. For the function $u \in C(U, \Omega)$, the CFD of order $r \in (m-1,\, m)$ with the lower limit zero is described as

$$^{c}D^r u(\zeta) = \frac{1}{\Gamma(m-r)} \int_0^\zeta (\zeta - \kappa)^{m-r-1} u^{(m)}(\kappa)\,\mathrm{d}\kappa, \quad \zeta > 0.$$

Consider the following nonlinear fractional system:

$$\begin{aligned} {}^{c}D^{r}u(\zeta) &= Bu(\zeta), \quad \zeta \in U = [0,\eta], \\ u(0) &= u_0, \qquad u'(0) = u_1, \end{aligned} \tag{2}$$

where $B : D(B) \subset \Omega \to \Omega$ is a closed and densely defined operator in a Banach space Ω. Applying the fractional integral of order r to equation (2) allows us to find the solution as follows:

$$u(\zeta) = u_0 + \frac{1}{\Gamma(r)} \int_0^{\zeta} (\zeta-\kappa)^{r-1} Bu(\kappa)\,\mathrm{d}\kappa.$$

Definition 3. (See [18].) The r-order cosine family $Q_r(\zeta)$ is classified as exponentially bounded if and only if there exist two constants, $N \geqslant 1$ and $\alpha \geqslant 0$ such that the following condition is satisfied:

$$\|Q_r(\zeta)\| \leqslant N\mathrm{e}^{\alpha\zeta}, \quad \zeta > 0.$$

Definition 4. (See [23].) The RLF $S_r : [0,\infty) \to L(\Omega)$ associated with Q_r is derived by

$$S_r(\zeta) = I^{r-1}Q_r(\zeta).$$

Lemma 1. (*See* [28].) *Assume that Ω is a Banach space. For each $\zeta \in [0,\eta]$ and $u \in \Omega$, one has*

$$\|S_r(\zeta)u\| \leqslant \frac{N\eta^{r-1}}{\Gamma(\eta)}\|u\|.$$

Definition 5. (See [18].) Assume that Λ is a BLO on the Banach space Ω, $\{Q_r(\zeta)\}_{\zeta\geqslant 0} \subset \Lambda(\Omega)$ is a strongly continuous r-order fractional cosine family, and B is the infinitesimal generator of $Q_r(\zeta)$. Then for $r \in (1,2]$, $\{Q_r(\zeta)\}_{\zeta\geqslant 0}$ is a solution operator of (2) if it satisfies the following conditions:

(i) $Q_r(\zeta)\lambda$ is a solution to (2) for $\lambda \in D(B)$;
(ii) $Q_r(\zeta)$ is strongly continuous for $\zeta \geqslant 0$, and $Q_r(0) = I$, where I is the identity mapping;
(iii) $Q_r(\zeta)D(B) \subset D(B)$, and for $\lambda \in D(B)$, we have $BQ_r(\zeta)\lambda = Q_r(\zeta)B\lambda$, $\zeta \geqslant 0$;
(iv) $\{P_r(\zeta)\}_{\zeta\geqslant 0}$ denotes the sine family directly associated with the strongly continuous r-order fractional cosine family $\{Q_r(\zeta)\}$, and is defined by $P_r(\zeta)u = \int_0^{\zeta} Q_r(\kappa)u(\kappa)\,\mathrm{d}\kappa$, $\zeta \geqslant 0$.

Following the approach adopted by Kexue et al. [12], Zoubida et al. [28], and Sivasankar et al. [25], we convert the control system (1) into an equivalent integral equation. Such a formulation plays a crucial role in defining the mild solution of the system.

Definition 6. A function $u(\cdot) \in C(U, \varOmega)$ is called a mild solution of problem (1) if it is the solution to the following integral equation:

$$\begin{aligned} u(\zeta) &= Q_r(\zeta)\big[u_0 - p(u)\big] + P_r(\zeta)\big[u_1 - q(u)\big] \\ &\quad + \int_0^\zeta S_r(\zeta - \tau)\hbar\left(\tau, u(\tau), \int_0^\tau \varpi\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau \\ &\quad + \int_0^\zeta S_r(\zeta - \tau)E\varphi(\tau)\,\mathrm{d}\tau + \sum_{0<\zeta_k<\zeta} S_r(\zeta - \zeta_k)J_k\big(u(\zeta_k)\big), \quad \zeta, \tau \in U. \end{aligned}$$

Let us now review the definition of approximate controllability pertaining problem (1) as follows:

Definition 7. The control system (1) is approximately controllable on U if the closure of the reachable set $\overline{W(\eta; u_0, u_1, \varphi)}$ is dense in $\varOmega$, i.e., $\overline{W(\eta; u_0, u_1, \varphi)} = \varOmega$ for every pair of states $u_0, u_1 \in \varOmega$ and some corresponding control $\varphi \in L^2(0, \eta, V)$, where

$$W(\eta; u_0, u_1) = \big\{W(\eta, u_0, u_1, \varphi)\colon\ \varphi \in L^2(U, V),\ U = [0, \eta]\big\}.$$

In [17], Mahmudov and McKibben introduced the concept of approximate controllability for the following linear control problem as a natural generalization of the notion of approximate controllability for linear first-order control systems:

$$\begin{aligned} {}^cD^q u(\zeta) &= -Bu(\zeta) + E\varphi(\zeta), \quad \zeta \in U = [0, \eta], \\ u(0) &= u_0, \qquad u'(0) = u_1. \end{aligned} \tag{3}$$

According to [28], for convenience in the subsequent analysis, the operators $\varTheta_0^\eta, W(\gamma, \varTheta_0^\eta) : \varOmega \to \varOmega$ associated with (3) are defined as

$$\varTheta_0^\eta = \int_0^\eta S_r(\eta - \tau)EE^*S_r^*(\eta - \tau)\,\mathrm{d}\tau, \qquad W\big(\gamma, \varTheta_0^\eta\big) = \big(\gamma I + \varTheta_0^\eta\big)^{-1},$$

where $\gamma > 0$, E^* and S_r^* are the adjoint of E and S_r, respectively. Based on [16], $\varTheta_0^\eta$ is a BLO, and $W(\gamma, \varTheta_0^\eta)$ is continuous operator with $\|W(\gamma, \varTheta_0^\eta)\| \leqslant 1/\gamma$.

In the end of this part, we recall Krasnoselskii's FPT [10], a result that will be applied subsequently to establish the existence of mild solutions for problem (1).

Theorem 1 [Krasnoselskii's FPT]. *Assume that $\varLambda$ is a nonempty, bounded, closed, and convex subset of the Banach space $\varOmega$, and that $\mathfrak{T}_1, \mathfrak{T}_2 : \varLambda \to \varOmega$ are given mappings such that*:

(i) $\mathfrak{T}_1$ *is a contraction,*
(ii) $\mathfrak{T}_2$ *is completely continuous,*
(iii) $\mathfrak{T}_1 u + \mathfrak{T}_2 v \in \varLambda$ *for any* $u, v \in \varLambda$.

Then the equation $\mathfrak{T}_1 u + \mathfrak{T}_2 u = u$ admits a solution on $\varLambda$.

3 Hypotheses

In this part, we list the necessary and essential assumptions that will help us accomplish the desired task in this manuscript.

- (A_1) The uniformly bounded linear cosine family $\{Q_r(\zeta)\}_{\zeta>0}$ generated by B is compact and continuous in the uniform operator topology for any $\zeta \in U = [0, \eta]$.
- (A_2) The function $\hbar : U \times \Omega^3 \to \Omega$, $\Omega^3 = \Omega \times \Omega \times \Omega$, fulfills the assertions below:
  - (i) For all $(u, v, w) \in \Omega^3$, $\hbar(\cdot, u, v, w)$ is measurable;
  - (ii) For a.e.$\zeta \in U$, $\hbar(\zeta, \cdot, \cdot, \cdot)$ is continuous;
  - (iii) There exist constants Q_1, Q_2, and Q_3 such that for all $(u, v, w) \in \Omega^3$ and $(\widetilde{u}, \widetilde{v}, \widetilde{w}) \in \Omega^3$,
  $$\big\|\hbar(\zeta, u, v, w) - \hbar(\zeta, \widetilde{u}, \widetilde{v}, \widetilde{w})\big\| \leqslant Q_1\|u - \widetilde{u}\| + Q_2\|v - \widetilde{v}\| + Q_3\|w - \widetilde{w}\|.$$
  - (iv) For all $\zeta \in U$, there exists a positive function $\psi \in L^\infty(U, \mathbb{R}^+)$ such that
  $$\left\|\hbar\left(\zeta, u(\zeta), \int_0^\zeta \varpi\big(\zeta, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\zeta, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)\right\| \leqslant \psi(\zeta).$$
- (A_3) The functions $\varpi, \sigma : U \times U \times \Omega \to \Omega$ fulfill the axioms below:
  - (i) For all $(\kappa, u) \in U \times \Omega$, $\hbar(\cdot, \kappa, u)$ and $\sigma(\cdot, \kappa, u)$ are measurable;
  - (ii) For a.e. $\zeta \in U$, $\hbar(\zeta, \cdot, \cdot)$ and $\sigma(\zeta, \cdot, \cdot)$ are continuous;
  - (iii) For all $\zeta, \kappa \in U$ and all $u, \widetilde{u} \in \Omega$, there exist positive constants Q_4 and Q_5 such that
  $$\big\|\varpi(\zeta, \kappa, u) - \varpi(\zeta, \kappa, \widetilde{u})\big\| \leqslant Q_4\|u - \widetilde{u}\|$$
  and
  $$\big\|\sigma(\zeta, \kappa, u) - \sigma(\zeta, \kappa, \widetilde{u})\big\| \leqslant Q_5\|u - \widetilde{u}\|.$$
- (A_4) The functions $p, q : \Omega \to \overline{D(B)}$ are continuous, and there exists constants Q_6 and Q_7 such that for all $u, \widetilde{u} \in \Omega$,
$$\big\|p(u) - p(\widetilde{u})\big\| \leqslant Q_6\|u - \widetilde{u}\| \quad \text{and} \quad \big\|q(u) - q(\widetilde{u})\big\| \leqslant Q_7\|u - \widetilde{u}\|.$$
- (A_5) The impulses $J_k : \Omega \to \Omega$ ($k = 1, 2, \dots, m$) are continuous, and there exists $\beta_k > 0$ such that for all $u \in \Omega$,
$$\big\|J_k(u)\big\| \leqslant \beta_k\|u\|.$$

Moreover, for any $\gamma \in (0, 1)$ and any $u_d \in \Omega$, we choose a proper control φ_γ such that there exists a mild solution $u(\cdot, u_0, u_1, \varphi_\gamma) \in (U, \Omega)$ for system (1) satisfying

$$\varphi_\gamma(\zeta, u) \\ = E^* S_r^*(\eta - \zeta)\big(\gamma I + \Theta_0^\eta\big)^{-1}\big[u_d - Q_r(\eta)\big(u_0 - p(u)\big) - P_r(\eta)\big(u_1 - q(u)\big)\big]$$

$$
\begin{aligned}
&- E^{*}S_r^{*}(\eta-\zeta)\int_0^{\eta}\big(\gamma I+\Theta_0^{\eta}\big)^{-1}S_r(\eta-\tau)\\
&\qquad\times\hbar\left(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\\
&- E^{*}S_r^{*}(\eta-\zeta)\big(\gamma I+\Theta_0^{\eta}\big)^{-1}\sum_{k=1}^{m}S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big),\quad \tau\in[0,\eta].
\end{aligned}\tag{4}
$$

4 Existence of mild solutions

In this part, we discuss the existence of a mild solution to the fractional problem (1).

According to the definition of control (4), define the operator $\mathfrak{T}: C(U,\Omega)\to C(U,\Omega)$ by

$$
\begin{aligned}
&(\mathfrak{T}u)(\zeta)\\
&\quad= Q_r(\zeta)\big[u_0-p(u)\big]+P_r(\zeta)\big[u_1-q(u)\big]\\
&\qquad+\int_0^{\zeta}S_r(\zeta-\tau)\hbar\left(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\\
&\qquad+\int_0^{\zeta}S_r(\zeta-\tau)E\varphi(\tau)\,\mathrm{d}\tau+\sum_{0<\zeta_k<\zeta}S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big),\quad \zeta,\tau\in U.
\end{aligned}
$$

Clearly, finding a mild solution of the fractional problem (1) is equivalent to finding a fixed point of the operator $\mathfrak{T}(\cdot)$. To this end, we use the Krasnoselskii's FPT. In order to apply Krasnoselskii's FPT, assume that $E_s=\{u\in C(U,\Omega)\colon \|u\|_C\leqslant s\}$ for each $s>0$. It is clear that E_s is a bounded, closed, and convex subset of $C(U,\Omega)$.

Next, we define the operators $\mathfrak{T}_1$ and $\mathfrak{T}_2$ on E_s as follows:

$$
\begin{aligned}
(\mathfrak{T}_1u)(\zeta)&=Q_r(\zeta)\big[u_0-p(u)\big]+P_r(\zeta)\big[u_1-q(u)\big],\\
(\mathfrak{T}_2u)(\zeta)&=\int_0^{\zeta}S_r(\zeta-\tau)\hbar\left(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\\
&\quad+\int_0^{\zeta}S_r(\zeta-\tau)E\varphi(\tau)\,\mathrm{d}\tau+\sum_{0<\zeta_k<\zeta}S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big).
\end{aligned}
$$

Now, we can state and prove our main theorem in this part.

Theorem 2. *Under hypotheses* (A_1)–(A_5)*, the fractional control problem* (1) *admits at least one mild solution on* U*, provided that* $\{S_r(\zeta)\}$ *is compact for every* $\zeta>0$ *and* $N(Q_6+Q_7)<1$.

Proof. We split the proof into the following steps:

Step 1. Prove that there exist constants $\Upsilon_1, \Upsilon_2 > 0$ such that

$$\left\|\varphi_\gamma(\zeta, u)\right\| \leqslant \frac{N_E}{\gamma}\left(\Upsilon_1 + \Upsilon_2\|u\|_C\right)$$

for all $\zeta \in U$ and all $u \in C(U, \Omega)$, where

$$\begin{aligned}
\Upsilon_1 &= \frac{N\eta^{r-1}}{\Gamma(\eta)}\left(\|u_d\| + N\Xi + \frac{N\eta^r}{\Gamma(\eta)}\|\psi\|_{L^\infty(U,\mathbb{R}^+)}\right),\\
\Xi &= \|u_0\| + \left\|p(u)\right\| + \|u_1\| + \left\|q(u)\right\|\\
\Upsilon_2 &= Q_1 + Q_2Q_4 + Q_3Q_5 + \sum_{k=1}^m \beta_k.
\end{aligned}$$

Based on (4) and the result of Lemma 1, we find that for $\zeta, \eta \in U$,

$$\begin{aligned}
&\left\|\varphi_\gamma(\zeta, u)\right\| \\
&\quad\leqslant \Bigg\| E^* S_r^*(\eta - \zeta)\left(\gamma I + \Theta_0^\eta\right)^{-1}\left[u_d - Q_r(\eta)\left(u_0 - p(u)\right) - P_r(\eta)\left(u_1 - q(u)\right)\right] \\
&\qquad - \int_0^\eta S_r(\eta - \tau)\hbar\left(\tau, u(\tau), \int_0^\tau \varpi\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa, \int_0^\eta \sigma\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa\right) \mathrm{d}\tau \\
&\qquad - \sum_{k=1}^m S_r(\zeta - \zeta_k) J_k\left(u(\zeta_k)\right)\Bigg\| \\
&\quad\leqslant \frac{N_E N\eta^{r-1}}{\gamma\Gamma(\eta)}\Bigg\| u_d - Q_r(\eta)\left(u_0 - p(u)\right) - P_r(\eta)\left(u_1 - q(u)\right) \\
&\qquad - \int_0^\eta S_r(\eta - \tau)\hbar\left(\tau, u(\tau), \int_0^\tau \varpi\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa, \int_0^\eta \sigma\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa\right) \mathrm{d}\tau \\
&\qquad - \sum_{k=1}^m S_r(\zeta - \zeta_k) J_k\left(u(\zeta_k)\right)\Bigg\| \\
&\quad\leqslant \frac{N_E N\eta^{r-1}}{\gamma\Gamma(\eta)}\Bigg\{\|u_d\| + N\|u_0 - p(u)\| + N\left\|u_1 - q(u)\right\| \\
&\qquad - \frac{N\eta^{r-1}}{\Gamma(\eta)}\Bigg[\int_0^\eta \Bigg\|\hbar\left(\tau, u(\tau), \int_0^\tau \varpi\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa, \int_0^\eta \sigma\left(\tau, \kappa, u(\kappa)\right) \mathrm{d}\kappa\right) \\
&\qquad\qquad - \hbar\left(\tau, 0, \int_0^\tau \varpi(\tau, \kappa, 0) \mathrm{d}\kappa, \int_0^\eta \sigma(\tau, \kappa, 0) \mathrm{d}\kappa\right)\Bigg\| \mathrm{d}\tau
\end{aligned}$$

$$+\left\|\int_0^\eta \hbar\left(\tau,0,\int_0^\tau \varpi(\tau,\kappa,0)\,\mathrm{d}\kappa,\int_0^\eta \sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\right\|\Bigg]$$
$$-\frac{N\eta^{r-1}}{\Gamma(\eta)}\sum_{k=1}^m \big\|J_k\big(u(\zeta_k)\big)\big\|\Bigg\}.$$

From our assumptions we have

$$\begin{aligned}
&\big\|\varphi_\gamma(\zeta,u)\big\| \\
&\quad\leqslant \frac{N_E N\eta^{r-1}}{\gamma\Gamma(\eta)}\Bigg\{\|u_d\|+N\big[\|u_0\|+\big\|p(u)\big\|\big]+N\big[\|u_1\|+\big\|q(u)\big\|\big] \\
&\qquad+\frac{N\eta^r}{\Gamma(\eta)}\Bigg[\int_0^\eta\left(Q_1\|u\|+Q_2Q_4\int_0^\tau\big\|u(\kappa)\big\|\,\mathrm{d}\kappa+Q_3Q_5\int_0^\eta\big\|u(\kappa)\big\|\,\mathrm{d}\kappa\right)\mathrm{d}\tau \\
&\qquad+\int_0^\eta\big\|\psi(\kappa)\big\|\,\mathrm{d}\kappa\Bigg]+\frac{N\eta^{r-1}}{\Gamma(\eta)}\sum_{k=1}^m\beta_k\|u\|\Bigg\} \\
&\quad\leqslant \frac{N_E N\eta^{r-1}}{\gamma\Gamma(\eta)}\Bigg\{\|u_d\|+N\big[\|u_0\|+\big\|p(u)\big\|+\|u_1\|+\big\|q(u)\big\|\big] \\
&\qquad+\frac{N\eta^{r-1}}{\Gamma(\eta)}\Bigg[Q_1\|u\|_C+Q_2Q_4\|u\|_C+Q_3Q_5\|u\|_C \\
&\qquad+\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\sum_{k=1}^m\beta_k\|u\|_C\Bigg]\Bigg\} \\
&\quad\leqslant \frac{N_E}{\gamma}\big(\varUpsilon_1+\varUpsilon_2\|u\|_C\big).
\end{aligned}$$

This completes the step.

Step 2. Claim that $\mathfrak{T}_1$ is a contraction on E_s. For any $\zeta\in U$ and $u,v\in E_s$, by applying assertion (A_4), we have

$$\begin{aligned}
&\big|(\mathfrak{T}_1u)(\zeta)-(\mathfrak{T}_1v)(\zeta)\big\| \\
&\quad\leqslant \big\|Q_r(\zeta)\big\|\big\|p(u)-p(v)\big\|+\big\|P_r(\zeta)\big\|\big\|q(u)-q(v)\big\| \\
&\quad\leqslant NQ_6\|u-v\|+NQ_7\|u-v\|\leqslant N(Q_6+Q_7)\|u-v\|.
\end{aligned}$$

Since $N(Q_6+Q_7)<1$, we conclude that $\mathfrak{T}_1$ is a contraction.

Step 3. We claim that $\mathfrak{T}_2$ is completely continuous. First, we illustrate that $\mathfrak{T}_2$ is continuous on E_s. Let $\{u_m\}\subset E_s$ and $u_m\to u$ in E_s. Applying hypotheses (A_2) and

(A_3) along with this convergence, it follows that

$$\hbar\left(\tau, u_m, \int_0^\tau \varpi\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau$$
$$\to \hbar\left(\tau, u, \int_0^\tau \varpi\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau \quad \text{as } m \to \infty,$$

and

$$\sum_{k=1}^m J_k\big(u_m(\zeta_k)\big) \to \sum_{k=1}^m J_k\big(u(\zeta_k)\big) \quad \text{as } m \to \infty.$$

Applying the dominated convergence theorem, we find that

$$\begin{aligned}
&\big\|(\mathfrak{T}_2 u_m)(\zeta) - (\mathfrak{T}_2 u)(\zeta)\big\| \\
&\quad\leqslant \frac{N\eta^{r-1}}{\Gamma(\eta)} \int_0^\eta \bigg\|\hbar\left(\tau, u_m(\tau), \int_0^\tau \varpi\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa\right) \\
&\qquad - \hbar\left(\tau, u(\tau), \int_0^\tau \varpi\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^{\eta'} \sigma\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)\bigg\|\mathrm{d}\tau \\
&\qquad + \frac{N_E}{\gamma}\left(\frac{N_E N\eta^{r-1}}{\Gamma(\eta)}\right)^2 \bigg\{\eta\big\|Q_r(\zeta)\big\|\big\|p(u_m) - p(u)\big\| + \eta\big\|P_r(\zeta)\big\|\big\|q(u_m) - q(u)\big\| \\
&\qquad + \frac{N\eta^r}{\Gamma(\eta)} \int_0^\zeta \bigg[\bigg\|\hbar\left(\delta, u_m, \int_0^\delta \varpi(\delta, \kappa, u_m)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\delta, \kappa, u_m)\,\mathrm{d}\kappa\right) \\
&\qquad\qquad - \hbar\left(\delta, u, \int_0^\delta \varpi(\delta, \kappa, u)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\delta, \kappa, u)\,\mathrm{d}\kappa\right)\bigg\| \\
&\qquad\qquad + \sum_{k=1}^m \big\|J_k\big(u_m(\zeta_k)\big) - J_k\big(u(\zeta_k)\big)\big\|\bigg]\,\mathrm{d}\delta\bigg\} \\
&\qquad + \frac{N\eta^r}{\Gamma(\eta)} \sum_{k=1}^m \big\|J_k\big(u_m(\zeta_k)\big) - J_k\big(u(\zeta_k)\big)\big\| \\
&\quad\leqslant \frac{N\eta^{r-1}}{\Gamma(\eta)} \int_0^\eta \bigg\|\hbar\left(\tau, u_m(\tau), \int_0^\tau \varpi\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u_m(\kappa)\big)\,\mathrm{d}\kappa\right) \\
&\qquad\qquad - \hbar\left(\tau, u(\tau), \int_0^\tau \varpi\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau, \kappa, u(\kappa)\big)\,\mathrm{d}\kappa\right)\bigg\|\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{N_E}{\gamma}\left(\frac{N_E N \eta^{r-1}}{\Gamma(\eta)}\right)^2 \Bigg\{ N(Q_6+Q_7)\|u_m - u\| \\
&+ \frac{N\eta^r}{\Gamma(\eta)}\Bigg[\int_0^\zeta \Bigg\| \hbar\Bigg(\delta, u_m, \int_0^\delta \varpi(\delta,\kappa,u_m)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\delta,\kappa,u_m)\,\mathrm{d}\kappa\Bigg) \\
&\qquad\qquad - \hbar\Bigg(\delta, u, \int_0^\delta \varpi(\delta,\kappa,u)\,\mathrm{d}\kappa, \int_0^{\delta'} \sigma(\delta,\kappa,u)\,\mathrm{d}\kappa\Bigg)\Bigg\|\,\mathrm{d}\delta \\
&+ 2\sum_{k=1}^m \big\|J_k\big(u_m(\zeta_k)\big) - J_k\big(u(\zeta_k)\big)\big\|\Bigg]\Bigg\} \\
&\to 0 \quad \text{as } m \to +\infty,
\end{aligned}
$$

where $\delta \in [0,\eta]$. Thus, $\mathfrak{T}_2$ is continuous on E_s. Next, we show that $\mathfrak{T}_2$ is a compact operator. To establish this, we show that the set $\{(\mathfrak{T}_2 u)(\zeta)\colon u \in E_s\}$ is relatively compact in Ω. This is followed by proving that the same set is uniformly bounded. For any $\zeta \in U$ and $u \in E_s$, from Lemma 1 and Step 1 we have

$$
\begin{aligned}
&\big\|(\mathfrak{T}_2 u)(\zeta)\big\| \\
&\leqslant \frac{N\eta^{r-1}}{\Gamma(\eta)}\Bigg(\int_0^\eta \Bigg\| \hbar\Bigg(\tau, u(\tau), \int_0^\tau \varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\Bigg) \\
&\qquad\qquad - \hbar\Bigg(\tau, 0, \int_0^\tau \varpi(\tau,\kappa,0)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\Bigg)\Bigg\|\,\mathrm{d}\tau \\
&\quad + \int_0^\eta \Bigg\|\hbar\Bigg(\tau, 0, \int_0^\tau \varpi(\tau,\kappa,0)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\Bigg)\Bigg\|\,\mathrm{d}\tau \\
&\quad + \int_0^\zeta \big\|E\varphi_\gamma(\tau,u)\big\|\,\mathrm{d}\tau + \sum_{k=1}^m \big\|J_k\big(u(\zeta_k)\big)\big\|\Bigg) \\
&\leqslant \frac{N\eta^r}{\Gamma(\eta)}\Bigg(Q_1\|u\|_C + Q_2Q_4\|u\|_C + Q_3Q_5\|u\|_C \\
&\quad + \|\psi\|_{L^\infty(U,\mathbb{R}^+)} + \frac{N_E}{\gamma}\big\|\varphi_\gamma(\tau,u)\big\| + \sum_{k=1}^m \beta_k\|u\|_C\Bigg) \\
&\leqslant \frac{N\eta^r}{\Gamma(\eta)}\Bigg(Q_1 s + Q_2Q_4 s + Q_3Q_5 s + \eta\|\psi\|_{L^\infty(U,\mathbb{R}^+)} + \frac{N_E}{\gamma}(\varUpsilon_1 + \varUpsilon_2 s) + s\sum_{k=1}^m \beta_k\Bigg).
\end{aligned}
$$

Hence, $\sup_{u\in E_s}\|(\mathfrak{T}_2u)\|<\infty$, which means that $\{\mathfrak{T}_2u\colon u\in E_s\}$ is uniformly bounded. Now, we claim that $\{\mathfrak{T}_2u\colon u\in E_s\}$ is an equicontinuous family of functions on $U=[0,\eta]$. For all $\zeta\in U$ and any fixed $u\in E_s$ with $0\leqslant\zeta_1<\zeta_2\leqslant\eta$, one has

$$
\begin{aligned}
&\big\|(\mathfrak{T}_2u)(\zeta_2)-(\mathfrak{T}_2u)(\zeta_1)\big\|\\
&\quad\leqslant\Bigg\|\int_0^{\zeta_2}S_r(\zeta_2-\tau)\hbar\Bigg(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\,\mathrm{d}\tau\\
&\qquad-\int_0^{\zeta_1}S_r(\zeta_1-\tau)\hbar\Bigg(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\,\mathrm{d}\tau\\
&\qquad+\int_0^{\zeta_2}S_r(\zeta_2-\tau)E\varphi(\tau)\,\mathrm{d}\tau-\int_0^{\zeta_1}S_r(\zeta_1-\tau)E\varphi(\tau)\,\mathrm{d}\tau\\
&\qquad+\sum_{0<\zeta_k<\zeta_2}S_r(\zeta_2-\zeta_k)J_k\big(u(\zeta_k)\big)-\sum_{0<\zeta_k<\zeta_1}S_r(\zeta_1-\zeta_k)J_k\big(u(\zeta_k)\big)\Bigg\|\\
&\quad\leqslant\int_0^{\zeta_1}\big\|S_r(\zeta_2-\tau)-S_r(\zeta_1-\tau)\big\|\\
&\qquad\times\Bigg\{\Bigg\|\hbar\Bigg(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\mathrm{d}\kappa\Bigg)\\
&\qquad-\hbar\Bigg(\tau,0,\int_0^{\tau}\varpi(\tau,\kappa,0)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\Bigg)\Bigg\|\\
&\qquad+\Bigg\|\hbar\Bigg(\tau,0,\int_0^{\tau}\varpi(\tau,\kappa,0)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\Bigg)\Bigg\|+\big\|E\varphi_\gamma(\tau,u)\big\|\Bigg\}\,\mathrm{d}\tau\\
&\qquad+\Bigg\|\sum_{0<\zeta_k<\zeta_1}\big[S_r(\zeta_2-\tau)-S_r(\zeta_1-\tau)\big]J_k\big(u(\zeta_k)\big)\Bigg\|\\
&\qquad+\int_{\zeta_1}^{\zeta_2}\big\|S_r(\zeta_2-\tau)\big\|\Bigg(\Bigg\|\hbar\Bigg(\tau,u(\tau),\int_0^{\tau}\varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa,\int_0^{\eta}\sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\Bigg\|\\
&\qquad+\big\|E\varphi_\gamma(\tau,u)\big\|\Bigg)\,\mathrm{d}\tau\\
&\qquad+\Bigg\|\sum_{\zeta_1<\zeta_k<\zeta_2}S_r(\zeta_2-\tau)J_k\big(u(\zeta_k)\big)\Bigg\|.
\end{aligned}
$$

It follows from Lemma 1, Step 1, and hypotheses (A_2)–(A_5) that

$$\begin{aligned}
&\big\|(\mathfrak{T}_2u)(\zeta_2)-(\mathfrak{T}_2u)(\zeta_1)\big\| \\
&\quad\leqslant \int_0^{\zeta_1}\big\|S_r(\zeta_2-\tau)-S_r(\zeta_1-\tau)\big\|\Big\{Q_1\|u\|_C+Q_2Q_4\|u\|_C+Q_3Q_5\|u\|_C \\
&\qquad\quad+\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E}{\gamma}\big(\Upsilon_1+\Upsilon_2\|u\|_C\big)\Big\}\,\mathrm{d}\tau \\
&\qquad+\sum_{0<\zeta_k<\zeta_1}\big\|S_r(\zeta_2-\tau)-S_r(\zeta_1-\tau)\big\|\big\|J_k\big(u(\zeta_k)\big)\big\| \\
&\qquad+\frac{N\eta^r}{\Gamma(\eta)}\int_{\zeta_1}^{\zeta_2}\Big\{Q_1\|u\|_C+Q_2Q_4\|u\|_C+Q_3Q_5\|u\|_C+\|\psi\|_{L^\infty(U,\mathbb{R}^+)} \\
&\qquad\qquad+\frac{N_E}{\gamma}\big(\Upsilon_1+\Upsilon_2\|u\|_C\big)\Big\}\,\mathrm{d}\tau \\
&\qquad+\frac{N\eta^r}{\Gamma(\eta)}\sum_{\zeta_1<\zeta_k<\zeta_2}\big\|J_k\big(u(\zeta_k)\big)\big\| \\
&\quad\leqslant \int_0^{\zeta_1}\big\|S_r(\zeta_2-\tau)-S_r(\zeta_1-\tau)\big\|\Big\{Q_1s+Q_2Q_4s+Q_3Q_5s \\
&\qquad\quad+\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E}{\gamma}(\Upsilon_1+\Upsilon_2s)\Big\}\,\mathrm{d}\tau \\
&\qquad+\frac{N\eta^r}{\Gamma(\eta)}\Big\{Q_1s+Q_2Q_4s+Q_3Q_5s+\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E}{\gamma}(\Upsilon_1+\Upsilon_2s)\Big\} \\
&\qquad\times|\zeta_2-\zeta_1|+\frac{N\eta^r s}{\Gamma(\eta)}\sum_{k=\zeta_1}^{\zeta_2}\beta_k \\
&\quad\to 0\quad\text{as }\zeta_2\to\zeta_1.
\end{aligned}$$

This follows because $S_r(\zeta)$ (for $\zeta>0$) is compact and uniformly continuous operator. This proves that the family of functions $\{\mathfrak{T}_2u\colon u\in E_s\}$ is equicontinuous. Consequently, the Arzelà–Ascoli theorem proves that $\mathfrak{T}_2$ is compact.

Step 4. Show that $\mathfrak{T}_1u+\mathfrak{T}_2v\in E_s$ for every $u,v\in E_s$. Using Lemma 1, Step 1, and assertions (A_2)–(A_5), one has

$$\begin{aligned}
&\big\|(\mathfrak{T}_1u)(\zeta)+(\mathfrak{T}_2v)(\zeta)\big\| \\
&\quad\leqslant \big\|Q_r(\zeta)\big\|\big\|u_0-p(u)\big\|+\big\|P_r(\zeta)\big\|\big\|u_1-q(u)\big\|
\end{aligned}$$

$$
\begin{aligned}
&+\frac{N\eta^{r-1}}{\Gamma(\eta)}\Biggl\{\int_0^\eta \Biggl\|\hbar\Biggl(\tau,u(\tau),\int_0^\tau \varpi\bigl(\tau,\kappa,u(\kappa)\bigr)\,\mathrm{d}\kappa,\int_0^\eta \sigma\bigl(\tau,\kappa,u(\kappa)\bigr)\,\mathrm{d}\kappa\Biggr) \\
&\qquad -\hbar\Biggl(\tau,0,\int_0^\tau \varpi(\tau,\kappa,0)\,\mathrm{d}\kappa,\int_0^\eta \sigma(\tau,\kappa,0)\,\mathrm{d}\kappa\Biggr)\Biggr\|\mathrm{d}\tau \\
&+\frac{N_E}{\gamma}\bigl(\varUpsilon_1+\varUpsilon_2\|u\|_C\bigr)+\sum_{k=1}^m\beta_k\|u\|\Biggr\} \\
&\leqslant N\bigl(\|u_0\|+\|p(u)\|+\|u_1\|+\|q(u)\|\bigr) \\
&+\frac{N\eta^r}{\Gamma(\eta)}\Biggl\{Q_1\|u\|_C+Q_2Q_4\|u\|_C+Q_3Q_5\|u\|_C \\
&+\eta\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E}{\gamma}\bigl(\varUpsilon_1+\varUpsilon_2\|u\|_C\bigr)+\sum_{k=1}^m\beta_k\|u\|\Biggr\} \\
&=N\Xi+\frac{N\eta^r}{\Gamma(\eta)}\Biggl(Q_1s+Q_2Q_4s+Q_3Q_5s \\
&+\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E}{\gamma}(\varUpsilon_1+\varUpsilon_2 s)+s\sum_{k=1}^m\beta_k\Biggr) \\
&\leqslant \varUpsilon+\frac{N\eta^r}{\Gamma(\eta)}\Biggl(Q_1+Q_2Q_4+Q_3Q_5+\frac{N_E}{\gamma}\varUpsilon_2+\sum_{k=1}^m\beta_k\Biggr)s,
\end{aligned}
$$

where

$$
\varUpsilon=N\Xi+\frac{N\eta^r}{\Gamma(\eta)}\biggl(\|\psi\|_{L^\infty(U,\mathbb{R}^+)}+\frac{N_E\varUpsilon_1}{\gamma}\biggr).
$$

Thus, if s is chosen to be sufficiently large $s>0$, we establish the bound $\|\mathfrak{T}_1u+\mathfrak{T}_2v\|\leqslant s$. This ensures that $\mathfrak{T}_1u+\mathfrak{T}_2v\in E_s$ whenever $u,v\in E_s$, where $\varUpsilon_1$, $\varUpsilon_2$, and Ξ are defined in Step 1. Thus, Theorem 1 guarantees the existence of a fixed point $u(\cdot)$ for the operator $\mathfrak{T}=\mathfrak{T}_1+\mathfrak{T}_2$ in E_s. Consequently, the control system (1) possesses at least one mild solution on the interval $U=[0,\eta]$. This successfully concludes the proof. □

5 Approximate controllability

This section is dedicated to the study of the approximate controllability of the control problem (1).

For this purpose, we assume that the linear control system (3) is approximately controllable and that the function $\hbar$ is uniformly bounded on its domain. Based on the results of [16], we impose the following hypothesis:

(A$_6$) $\gamma W(\gamma,\Theta_0^\eta)=\gamma(\gamma I+\Theta_0^\eta)^{-1}\to 0$ as $\gamma\to+\infty$ in the strong operator topology.

Now, the main theorem in this section is as follows:

Theorem 3. *Assume that hypotheses* (A_1)–(A_6) *hold and that* $Q_r(\zeta)$ *and* $P_r(\zeta)$ *are compact for every* $\zeta > 0$. *Then the control problem* (1) *is approximately controllable on* U.

Proof. Let $u_\gamma(\cdot) = u(\cdot, u_0, u_1, \varphi_\gamma)$ be a fixed point of $\mathfrak{T}_1 + \mathfrak{T}_2$ in E_s. Thus, Theorem 2 establishes $u_\gamma(\cdot)$ as a mild solution of (1), which is generated by the control

$$\begin{aligned}
&\varphi_\gamma(\zeta, u)\\
&\quad = E^* S_r^*(\eta-\zeta)\big(\gamma I + \Theta_0^\eta\big)^{-1}\big[u_d - Q_r(\eta)\big(u_0 - p(u)\big) - P_r(\eta)\big(u_1 - q(u)\big)\big]\\
&\qquad - E^* S_r^*(\eta-\zeta)\int_0^\eta \big(\gamma I + \Theta_0^\eta\big)^{-1} S_r(\eta-\tau)\\
&\qquad\qquad \times \hbar\left(\tau, u(\tau), \int_0^\tau \varpi\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa, \int_0^{\eta'} \sigma\big(\tau,\kappa,u(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\\
&\qquad - E^* S_r^*(\eta-\zeta)\big(\gamma I + \Theta_0^\eta\big)^{-1}\sum_{k=1}^m S_r(\zeta-\zeta_k) J_k\big(u(\zeta_k)\big).
\end{aligned}$$

Taking $\zeta = \eta$, we obtain

$$\begin{aligned}
u_\gamma(\eta) &= Q_r(\eta)\big(u_0 - p(u_\gamma)\big) + P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&\quad + \int_0^\eta S_r(\eta-\tau)\hbar\left(\tau, u_\gamma(\tau), \int_0^\tau \varpi\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau\\
&\quad + \int_0^\eta S_r(\eta-\tau)\Bigg\{BE^* S_r^*(\eta-\zeta) W\big(\gamma, \Theta_0^\eta\big)\Bigg[u_d - Q_r(\eta)\big(u_0 - p(u_\gamma)\big)\\
&\qquad - P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&\qquad - \int_0^\eta S_r(\eta-\delta)\hbar\left(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\delta\Bigg]\\
&\qquad - \sum_{k=1}^m S_r(\zeta-\zeta_k) J_k\big(u_\gamma(\zeta_k)\big)\Bigg\}\,\mathrm{d}\tau\\
&\quad + \sum_{0<\zeta_k<\zeta} S_r(\zeta-\zeta_k) J_k\big(u(\zeta_k)\big)\\
&= Q_r(\eta)\big(u_0 - p(u_\gamma)\big) + P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&\quad + \int_0^\eta S_r(\eta-\tau)\hbar\left(\tau, u_\gamma(\tau), \int_0^\tau \varpi\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&+ \Theta_0^\eta W\big(\gamma, \Theta_0^\eta\big)\bigg\{u_d - Q_r(\eta)\big(u_0 - p(u_\gamma)\big) - P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&- \int_0^\eta S_r(\eta-\delta)\Bigg[\hbar\Bigg(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\mathrm{d}\delta\\
&\qquad - \sum_{k=1}^m S_r(\zeta-\zeta_k)J_k\big(u_\gamma(\zeta_k)\big)\Bigg]\mathrm{d}\delta\\
&+ \sum_{0<\zeta_k<\zeta} S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big)\bigg\}.
\end{aligned}
$$

From $(I-\Theta_0^\eta)W(\gamma,\Theta_0^\eta) = \gamma W(\gamma,\Theta_0^\eta)$ it follows that

$$
\begin{aligned}
u_\gamma(\eta) &= Q_r(\eta)\big(u_0 - p(u_\gamma)\big) + P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&\quad + \int_0^\eta S_r(\eta-\tau)\hbar\Bigg(\tau, u_\gamma(\tau), \int_0^\tau \varpi\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\tau,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\mathrm{d}\tau\\
&\quad + \big(-\gamma I+\gamma I+\Theta_0^\eta\big)W\big(\gamma,\Theta_0^\eta\big)\bigg\{u_d - Q_r(\eta)\big(u_0-p(u_\gamma)\big) - P_r(\eta)\big(u_1-q(u_\gamma)\big)\\
&\quad - \int_0^\eta S_r(\eta-\delta)\Bigg[\hbar\Bigg(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\\
&\qquad - \sum_{k=1}^m S_r(\zeta-\zeta_k)J_k\big(u_\gamma(\zeta_k)\big)\Bigg]\mathrm{d}\delta\\
&\quad + \sum_{0<\zeta_k<\zeta} S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big)\bigg\}\\
&= u_d - \gamma W\big(\gamma,\Theta_0^\eta\big)\bigg\{u_d - Q_r(\eta)\big(u_0 - p(u_\gamma)\big) - P_r(\eta)\big(u_1 - q(u_\gamma)\big)\\
&\quad - \int_0^\eta S_r(\eta-\delta)\hbar\Bigg(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\Bigg)\mathrm{d}\delta\\
&\quad - \sum_{k=1}^m S_r(\zeta-\zeta_k)J_k\big(u_\gamma(\zeta_k)\big)\bigg\}\\
&= u_d - \gamma W\big(\gamma,\Theta_0^\eta\big)H(u_\gamma),
\end{aligned}
$$

where

$$\begin{aligned} H(u_\gamma) &= u_d - Q_r(\eta)\big(u_0 - p(u_\gamma)\big) - P_r(\eta)\big(u_1 - q(u_\gamma)\big) \\ &\quad - \int_0^\eta S_r(\eta-\delta)\hbar\left(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right)\mathrm{d}\delta \\ &\quad - \sum_{k=1}^m S_r(\zeta-\zeta_k)J_k\big(u_\gamma(\zeta_k)\big). \end{aligned}$$

From (A_1)–(A_6) we have that the functions $\hbar$, ϖ, σ, and J_k ($k = 1,2,\dots,m$) are uniformly bounded on U. Hence, the sequence

$$\left\{\hbar\left(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right)\right\}$$

is uniformly bounded in U. Consequently, by passing to a subsequence if necessary, we may assume that it converges weakly to $\hbar'$ in $L^2(U,\Omega)$, and that $J_k(u_\gamma)$ converges weakly to $J_k(u)$ in $L^2(U,\Omega)$. It is evident that

$$\begin{aligned} \phi &= u_d - Q_r(\eta)\big(u_0 - p(u)\big) - P_r(\eta)\big(u_1 - q(u)\big) \\ &\quad - \int_0^\eta S_r(\eta-\delta)\hbar'\,\mathrm{d}\delta - \sum_{k=1}^m S_r(\zeta-\zeta_k)J_k\big(u(\zeta_k)\big). \end{aligned}$$

Therefore, for $\delta \in [0,\eta]$,

$$\begin{aligned} &\big\|H(u_\gamma) - \phi\big\| \\ &\quad = \left\|\int_0^\zeta S_r(\eta-\delta)\left[\hbar\left(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right) - \hbar'\right]\mathrm{d}\delta\right\| \\ &\qquad + \left\|\sum_{k=1}^m S_r(\zeta-\zeta_k)\big[J_k\big(u(\zeta_k)\big) - J_k\big(u(\zeta_k)\big)\big]\right\| \\ &\quad \leqslant \sup_{\zeta\in[0,\eta]}\left\|\int_0^\zeta S_r(\eta-\delta)\left[\hbar\left(\delta, u_\gamma(\delta), \int_0^\delta \varpi\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa, \int_0^\eta \sigma\big(\delta,\kappa,u_\gamma(\kappa)\big)\,\mathrm{d}\kappa\right) - \hbar'\right]\mathrm{d}\delta\right\| \\ &\qquad + \sup_{\zeta\in[0,\eta]}\left\|\sum_{k=1}^m S_r(\zeta-\zeta_k)\big[J_k\big(u(\zeta_k)\big) - J_k\big(u(\zeta_k)\big)\big]\right\|. \end{aligned} \tag{5}$$

The compactness property of the operator $S_r(\zeta)$ for every $\zeta > 0$ leads directly to the limit $\|H(u_\gamma) - \phi\| \to 0$ as $\gamma \to 0$. Moreover, based on Ed. (1), we derive

$$\begin{aligned} \big\|u_\gamma(\eta) - u_\gamma\big\| &= \big\|\gamma W\big(\gamma, \varTheta_0^\eta\big)(\phi)\big\| + \big\|\gamma W\big(\gamma, \varTheta_0^\eta\big)\big\|\big\|H(u_\gamma) - \phi\big\| \\ &\leqslant \big\|\gamma W\big(\gamma, \varTheta_0^\eta\big)(\phi)\big\| + \big\|H(u_\gamma) - \phi\big\|. \end{aligned}$$

Hypothesis (A_6) ensures that the approximate controllability of (3) is equivalent to the strong convergence $\gamma W(\gamma, \Theta_0^\eta) \to 0$ as $\gamma \to 0^+$. Drawing upon (A_6) and estimation (5), this convergence proves that $\|u_\gamma(\eta) - u_d\| \to 0$ as $\gamma \to 0^+$, thereby guaranteeing the approximate controllability of problem (1). This completes the proof. □

6 Application under illustrative examples

In this section, we provide two examples:

(i) The first example validates the functions appearing in hypotheses (A_2)–(A_4).

(ii) The second example demonstrates the existence of a solution for a fractional control system, serving as an application.

Example 1. Assume that $\Omega = L^2([0,\pi])$, $U = [0,\eta]$. Define the functions $\varpi : U \times U \times \Omega \to \Omega$, $\sigma : U \times U \times \Omega \to \Omega$, $\hbar : U \times \Omega^3 \to \Omega$, $J_k : U \times \Omega \to \Omega$, $p : \Omega \to \Omega$, and $q : \Omega \to \Omega$ by

$$\varpi(\zeta,\kappa,u) = \frac{\mathrm{e}^{-\kappa}}{4 + |u(\kappa,g)|}, \quad \sigma(\zeta,\kappa,u) = \frac{\mathrm{e}^{-2\kappa}}{9 + |u(\kappa,g)|}, \quad g \in [0,\pi],$$

$$\hbar\left(\zeta, u, \int_0^\zeta \varpi(\zeta,\kappa,u)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\zeta,\kappa,u)\,\mathrm{d}\kappa\right)$$

$$= \frac{\mathrm{e}^{-\zeta}|u(\zeta,g)|}{(4+\mathrm{e}^{-\zeta})(1+|u(\zeta,g)|)} + \int_0^\zeta \frac{\mathrm{e}^{-\kappa}}{2+|u(\kappa,g)|}\,\mathrm{d}\kappa + \int_0^\eta \frac{\mathrm{e}^{-2\kappa}}{9+|u(\kappa,g)|}\,\mathrm{d}\kappa,$$

$$J_k\big(u(\zeta_k)\big) = \ln(k) u(\zeta_k), \quad k = 1,2,\dots m,$$

$$p(u)(g) = \sum_{j=1}^t \frac{\vartheta_j}{2} u(\zeta)(g), \qquad q(u)(g) = \sum_{j=1}^t \frac{\theta_j}{3} u(\zeta)(g),$$

respectively, where t is a positive integer, $\vartheta_j, \theta_j > 0$. Furthermore, assume that there exist positive constants x_1 and x_2 such that $\sum_{j=1}^t |\vartheta_j| \leqslant x_1$ and $\sum_{j=1}^t |\theta_j| \leqslant x_2$. Now, we check our assumptions as follows:

- It is clear that the function $\hbar$ is measurable and continuous. For $u, u' \in \Omega$, we have

$$\left\| \hbar\left(\zeta, u, \int_0^\zeta \varpi(\zeta,\kappa,u)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\zeta,\kappa,u)\,\mathrm{d}\kappa\right) - \hbar\left(\zeta, u', \int_0^\zeta \varpi(\zeta,\kappa,u')\,\mathrm{d}\kappa, \int_0^\eta \sigma(\zeta,\kappa,u')\,\mathrm{d}\kappa\right) \right\|$$

$$\leqslant \frac{\mathrm{e}^{-\zeta}}{4+\mathrm{e}^{-\zeta}}\|u-u'\| + \frac{\mathrm{e}^{-\eta}}{2}\|u-u'\| + \frac{\mathrm{e}^{-2\eta}}{9}\|u-u'\|.$$

Thus, $Q_1 = \mathrm{e}^{-\zeta}/(4+\mathrm{e}^{-\zeta})$, $Q_2 = \mathrm{e}^{-\eta}/2$, and $Q_3 = \mathrm{e}^{-2\eta}/9$ for all $\zeta \in [0,\eta]$. Moreover,

$$\begin{aligned}&\left\|\hbar\left(\zeta, u, \int_0^{\zeta} \varpi(\zeta,\kappa,u)\,\mathrm{d}\kappa, \int_0^{\eta} \sigma(\zeta,\kappa,u)\,\mathrm{d}\kappa\right)\right\| \\ &\quad\leqslant \left\|\frac{\mathrm{e}^{-\zeta}}{(4+\mathrm{e}^{-\zeta})} + \frac{\mathrm{e}^{-\eta}}{2} + \frac{\mathrm{e}^{-2\eta}}{9}\right\| \leqslant \frac{1}{4+\mathrm{e}^{-\zeta}} + \frac{1}{2\mathrm{e}^{\zeta}} + \frac{1}{9\mathrm{e}^{-2\eta}} = \psi(\zeta)\end{aligned}$$

for $\psi \in L^2([0,\pi])$. Hence, condition (A_2) holds.

• From the definition of the functions ϖ and σ we find that ϖ and σ are measurable and continuous. Moreover, for $u, u' \in \Omega$, one has

$$\begin{aligned}\left\|\varpi(\zeta,\kappa,u) - \varpi(\zeta,\kappa,u')\right\| &= \left\|\frac{\mathrm{e}^{-\kappa}}{4+|u(\kappa,g)|} - \frac{\mathrm{e}^{-\kappa}}{4+|u'(\kappa,g)|}\right\| \\ &\leqslant \frac{e}{4}\|u-u'\|\end{aligned}$$

and

$$\begin{aligned}\left\|\sigma(\zeta,\kappa,u) - \sigma(\zeta,\kappa,u')\right\| &= \left\|\frac{\mathrm{e}^{-2\kappa}}{9+|u(\kappa,g)|} - \frac{\mathrm{e}^{-2\kappa}}{9+|u(\kappa,g)|}\right\| \\ &\leqslant \frac{\mathrm{e}^2}{9}\|u-u'\|.\end{aligned}$$

Thus, $Q_4 = e/4$ and $Q_5 = \mathrm{e}^2/9$. This validate hypothesis (A_3).

• Assume that $\overline{D(B)} = L^2([0,\pi])$. Clearly, the functions $p, q : \Omega \to \overline{D(B)}$ are continuous, and for all $u, u' \in \Omega$, we can write

$$\begin{aligned}\left\|p(u) - p(u')\right\| &= \left\|\sum_{j=1}^{t} \frac{\vartheta_j}{2} u(\zeta) - \sum_{j=1}^{t} \frac{\vartheta_j}{2} u'(\zeta)\right\| \leqslant \sum_{j=1}^{t} \frac{|\vartheta_j|}{2}\|u-u'\| \\ &\leqslant \frac{x_1}{2}\|u-u'\|\end{aligned}$$

and

$$\begin{aligned}\left\|q(u) - q(u')\right\| &= \left\|\sum_{j=1}^{t} \frac{\theta_j}{3} u(\zeta) - \sum_{j=1}^{t} \frac{\theta_j}{3} u'(\zeta)\right\| \leqslant \sum_{j=1}^{t} \frac{|\theta_j|}{3}\|u-u'\| \\ &\leqslant \frac{x_2}{3}\|u-u'\|.\end{aligned}$$

Thus, $Q_6 = x_1/2$ and $Q_7 = x_2/3$ for any $x_1, x_2 > 0$. This fulfills assumption (A_4).

• It is obvious that the impulses $J_k : \Omega \to \Omega$ $(k = 1, 2, \dots, m)$ are continuous, and for all $u \in \Omega$, we get

$$\left\|J_k(u)\right\| = \left\|\ln(k) u(\zeta_k)\right\| \leqslant k\|u\|.$$

Thus, $\beta_k = k > 0$. Then assertion (A_5) is true.

Example 2. Consider the following fractional control system:

$$
\begin{aligned}
&{}^{c}D^{r}\rho(\zeta,g) = \frac{\partial^2}{\partial g^2}\rho(\zeta,g) + \frac{\mathrm{e}^{-\zeta}|u(\zeta,g)|}{(4+\mathrm{e}^{-\zeta})(1+|u(\zeta,g)|)} + \int\limits_0^{\zeta} \frac{\mathrm{e}^{-\kappa}}{2+|u(\kappa,g)|}\,\mathrm{d}\kappa \\
&\qquad + \int\limits_0^{\eta} \frac{\mathrm{e}^{-2\kappa}}{9+|u(\kappa,g)|}\,\mathrm{d}\kappa\, E\varphi(\zeta,g), \quad \zeta \in U = [0,1],\ g \in [0,\pi], \\
&\rho(\zeta,0) = \rho(\zeta,\pi) = 0, \qquad \rho'(\zeta,0) = \rho'(\zeta,\pi) = 0, \\
&\rho(\zeta,0) + \sum_{j=1}^{t} \frac{\vartheta_j}{2}\rho(\zeta_j,g) = \rho_0(g), \quad \rho'(\zeta,0) + \sum_{j=1}^{t} \frac{\theta_j}{3}\rho(\zeta_j,g) = \rho_1(g), \quad g \in [0,\pi], \\
&\bigl[\rho\bigl(\zeta_k^{+},g\bigr) - \rho\bigl(\zeta_k^{-},g\bigr)\bigr] = J_k\bigl(\rho(\zeta_k,g)\bigr), \quad k = 1,2,\dots,m,\ g \in [0,\pi],
\end{aligned}
\tag{6}
$$

where ${}^{c}D^{q}$ is the CFD of order $q \in (1,2)$, $\vartheta_j, \theta_j \in \mathbb{R}$, $t \in \mathbb{N}$. Also, assume that there exist $x_1, x_2 > 0$ such that $\sum_{j=1}^{t} |\vartheta_j| \leqslant x_1$ and $\sum_{j=1}^{t} |\theta_j| \leqslant x_2$.

Let $\Omega = L^2([0,\pi])$. Define the mapping $B : D(B) \subset \Omega \to \Omega$ by $B(\rho) = -\partial^2(\rho)/\partial g^2$ with the domain

$$
D(B) = \bigl\{\rho(\cdot) \in \Omega\colon\ \rho, \rho' \text{ are absolutely continuous},\ \rho'' \in \Omega,\ \rho(0) = \rho(\pi) = 0\bigr\}.
$$

It is well known that the operator $-B$ is densely defined in the Banach space Ω, and it is the infinitesimal generator of a uniformly bounded strongly continuous cosine family $\{Q(\zeta)\}_{\zeta \geqslant 0}$.

Moreover, the operator $-B$ possesses a discrete spectrum defined by the eigenvalues $\mu_m = -m^2\pi^2$ and the corresponding normalized eigenvectors $e_m(g) = \sqrt{2/\pi}\sin(m\pi g)$ for $g \in [0,\pi]$ and $m \in \mathbb{N}$.

Obviously, $\{e_m\}_{m\in\mathbb{N}}$ forms an orthonormal basis for Ω. The operator B can then be expressed as the spectral decomposition

$$
B\rho = -\sum_{m=1}^{\infty} \mu_m \langle \rho, e_m\rangle e_m, \quad \rho \in D(B).
$$

The corresponding cosine and sine functions are given by

$$
Q(\zeta)\rho = \sum_{m=1}^{\infty} \cos(\sqrt{\mu_m}\zeta)\langle \rho, e_m\rangle e_m, \quad \rho \in \Omega,
$$

and

$$
P(\zeta)\rho = \sum_{m=1}^{\infty} \frac{1}{\sqrt{\mu_m}} \sin(\sqrt{\mu_m}\zeta)\langle \rho, e_m\rangle e_m, \quad \rho \in \Omega.
$$

It is straightforward to verify that for every $\zeta > 0$, the operator $P(\zeta)$ is compact, and for all $\zeta \geqslant 0$, $Q(\zeta)$ is strongly continuous with the uniform bound $\|Q(\zeta)\| \leqslant 1$.

Now, problem (6) is equivalent to the nonlocal abstract problem (1) in Ω with $u(\zeta)(g) = \rho(\zeta, g)$, the functions $\varpi, \sigma : U \times U \times \Omega \to \Omega$ are described as

$$\varpi(\zeta, \kappa, u) = \frac{\mathrm{e}^{-\kappa}}{4 + |u(\kappa, g)|}, \quad \sigma(\zeta, \kappa, u) = \frac{\mathrm{e}^{-2\kappa}}{9 + |u(\kappa, g)|}, \quad g \in [0, \pi],$$

the function $\hbar : U \times \Omega^3 \to \Omega$ is obtained by

$$\begin{aligned}
&\hbar\left(\zeta, u, \int_0^\zeta \varpi(\zeta, \kappa, u)\,\mathrm{d}\kappa, \int_0^\eta \sigma(\zeta, \kappa, u)\,\mathrm{d}\kappa\right)(g) \\
&\quad = \frac{\mathrm{e}^{-\zeta}|u(\zeta, g)|}{(4+\mathrm{e}^{-\zeta})(1+|u(\zeta, g)|)} + \int_0^\zeta \frac{\mathrm{e}^{-\kappa}}{2+|u(\kappa, g)|}\,\mathrm{d}\kappa + \int_0^\eta \frac{\mathrm{e}^{-2\kappa}}{9+|u(\kappa, g)|}\,\mathrm{d}\kappa, \quad g \in [0, \pi],
\end{aligned}$$

$$p(u)(g) = \sum_{j=1}^{t} \frac{\vartheta_j}{2} u(\zeta)(g), \qquad q(u)(g) = \sum_{j=1}^{t} \frac{\theta_j}{3} u(\zeta)(g),$$

and

$$J_k\big(u(\zeta_k)\big)(g) = J_k\big(\rho(\zeta_k, g)\big) = \big[\rho\big(\zeta_k^+, g\big) - \rho\big(\zeta_k^-, g\big)\big], \quad k = 1, 2, \dots, m.$$

It has been rigorously established that assumptions (A_2)–(A_5) hold. Let $\digamma$ be an infinite-dimensional space given by

$$\digamma = \left\{\varphi = \sum_{j=2}^{\infty} \langle\varphi_j, e_j\rangle e_j(g)\colon\ \sum_{j=2}^{\infty} \langle\varphi_j, e_j\rangle^2 < \infty\right\},$$

endowed with the norm $\|\varphi\| = \sqrt{\sum_{j=2}^{\infty} \langle\varphi_j, e_j\rangle^2}$, and let $E : \digamma \to \Omega$ be a linear continuous mapping described as

$$E\varphi = 2\varphi_1 e_1(g) + \sum_{j=2}^{\infty} \varphi_j e_j(g).$$

Hence,

$$\varphi(\zeta, u) = \sum_{j=2}^{\infty} \langle\varphi_j, e_j\rangle e_j(g) \in L^2(U, \digamma),$$

and

$$E\varphi(\zeta) = 2\varphi_1(\zeta) e_1(g) + \sum_{j=2}^{\infty} \big\langle\varphi_j(\zeta), e_j\big\rangle e_j(g) \in L^2(U, \Omega).$$

Clearly, $\|E\|$ is bounded, and assertion (A_6) holds. Hence, all auxiliary assumptions of Theorem 3 are satisfied. Then the fractional control problem (6) is approximately controllable on $[0, 1]$.

7 Conclusion

This paper investigates the approximate controllability of a class of fractional control DEs. The considered system is characterized by the inclusion of CFDs, Volterra–Fredholm integral terms, impulsive effects, and nonlocal conditions of order $r \in (1,2)$ in the setting of Banach spaces. The main objective is to establish sufficient conditions guaranteeing the approximate controllability of the proposed control problem. Assuming that the associated linear system is approximately controllable, the analysis is developed by employing techniques from fractional calculus, Krasnoselskii's FPT, and the theory of resolvent operators. The obtained results extend and improve several related contributions available in the existing literature. To demonstrate the applicability of the theoretical findings, a concrete example is presented. The present study also involves a number of mathematical and practical challenges. The memory and hereditary nature of fractional derivatives substantially increase the complexity of the analysis when compared with classical integer-order systems. Moreover, the coexistence of Volterra and Fredholm integral operators produces both local and global interactions, which make the verification of compactness, continuity, and fixed-point assumptions more subtle. The nonlocal conditions add further difficulty since the initial state depends on the entire history of the solution, while impulsive effects introduce discontinuities that require suitable piecewise analytical frameworks. Future research may address extensions to stochastic and uncertain systems, variable-order and distributed-order fractional models, time-delay equations with state-dependent impulses, as well as efficient numerical schemes and computational methods for applications arising in engineering, biology, and networked control systems.

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