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Event-triggered distributed fixed-time optimization for multi-agent system with input time delays over directed networks*

Ruiqi Geng[a], **Qiang Zhang**[a], **Fei Wang**[a], **Ning Li**[b]

[a] School of Mathematical Sciences, Qufu Normal University,
Qufu Shandong, 273165, China
fei_9206@163.com

[b] College of Mathematics and Information Science,
Henan University of Economics and Law,
Zhengzhou Henan, 450046, China

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Abstract. This paper develops an event-triggered distributed optimization algorithm (DOA) for directed multi-agent systems. The algorithm is designed to solve fixed-time optimization problems with equality constraints, while also compensating for input time delays. Firstly, the time-delay system is transformed into an equivalent delay-free model by using the method of model order reduction. Secondly, to reduce the consumption of communication resources, an event-triggered communication scheme is utilized. Subsequently, based on Lyapunov stability theory, it is proved that the designed DOA can converge to the optimal solution within a fixed time. Finally, a numerical simulation shows the effectiveness and advantages of the event-triggered DOA for multi-agent systems.

Keywords: distributed optimization, fixed time, event-triggered communication scheme, input time delay.

1 Introduction

Optimization problems hold a central position in numerous scientific and engineering disciplines. Traditionally, these problems were solved using centralized algorithms [11, 19, 20]. However, in large-scale networks, where communication resources are limited and data cannot be centrally processed, centralized algorithms have become increasingly less applicable. For this reason, distributed optimization algorithms (DOAs) emerged as an important solution [4, 12, 21], which mainly includes distributed subgradient method, alternating direction method of multipliers, Newton method, and so on. Those methods

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address the overarching problem by breaking it down into a range of interconnected subproblems, which are addressed in a coordinated manner. Each agent only has local information about the data and communicates only with the neighboring nodes in the network, without the need for large-scale data transmission and reducing the communication burden.

In fact, the methods mentioned above are all asymptotically convergent. In practical applications, the primary objective is to achieve an optimal solution within a finite time. Because of this, discontinuous finite-time schemes were devised to address the convex optimization problems (CVOPs) in prior work [5, 28]; subsequently, a continuous finite-time algorithm was developed to solve the unconstrained optimization problem discussed in [17]. However, in these finite-time protocols, the estimated time depends on the initial conditions of the agents. To overcome this limitation, fixed-time DOA has garnered significant research interest. The estimated time is bounded above by a constant that is independent of the initial conditions, which means that regardless of the initial value, it will reach convergence within a specified time interval. The definition of fixed-time convergence was introduced and proven in [15]. Subsequent research has expanded this framework into DOA across different network settings. For example, a two-stage fixed-time algorithm employing dynamic event-triggering was designed in [25] for multi-agent systems (MASs) with strongly convex objectives; two fixed-time distributed adaptive algorithms were proposed in [9] to address a family of CVOPs for MASs. Further developments include an edge-based fixed-time protocol in [13], which can deal with both time-varying and time-invariant objective functions, as well as a continuous fixed-time method in [1] developed for equality constrained optimization. Although [1, 13] proposed fixed-time protocols to solve the distributed optimization problem (DOP), they were all applicable to undirected networks. In [2], a fixed-time DOA was introduced to solve the economic dispatch problem (EDP) for directed graphs; similarly, in [24], a DOA was proposed to address the DOP with communication delays and external disturbances in a directed network. In conclusion, the current work has made significant progress within the domain of distributed optimization at fixed times. From the construction of basic theories to the creation of algorithms for convex optimization, equality constraints, and so on, and successfully expanding the application from undirected graphs to more general directed graph networks, the related algorithms ensure the convergence of the system within fixed time.

However, most of the above research assumes that the information exchange between agents is completed instantaneously. In practical work, due to the nonideal data transmission and processing after reception, time delays are inevitable and have a substantial impact on the stability of the system [8, 22]. So dealing with time delays has become an important research topic. For instance, in [7], by incorporating virtual agents and redesigning communication links, desired delay properties were attained, enabling the reuse of established DOA to address the DOP. On the other hand, a DOA aimed at time-delay systems was developed in [26].

It should be noted that the aforementioned approaches necessarily involve continuous communication and controller updates, which may lead to energy waste. To avoid these above issues, the event-triggered method has been proposed [3, 10, 14]. By only exercising control at critical moments, the overall energy consumption of the system is optimized. Additionally, the event-triggered method prevents all agents from transmitting data simultaneously, effectively avoiding network congestion and data collisions, while enhancing reliability and real-time performance. Leveraging an event-triggered mechanism, a distributed primal-dual protocol was designed for dealing with a set of CVOPs in prior work [6]; an event-triggered DOA in [16] involving two stages was designed to address a fixed-time consensus-constrained DOP, where each agent possesses a strongly convex local objective function; within undirected networks, the event-triggered method was presented in [29] to address the DOP subject to equality constraints; in [27], an event-triggered DOA was designed to investigate the DOP subject to equality constraints; as reported in [18], leveraging an event-triggered mechanism, a DOA was also designed to address the DOP; more recently in [23], to address the issue of inadequate power supply, a fixed-time DOA, which employs an event-triggered mechanism, was developed. Existing research has achieved a series of results in event-triggered DOP. However, there remains potential for further extension. First, unlike undirected networks, directed graphs yield asymmetric Laplacian matrices, which complicates the construction of Lyapunov functions and the derivation of a uniform convergence time bound that is independent of initial conditions. Then the presence of input delays makes the control design more challenging. A model order reduction technique must be carefully applied to transform the delayed system into an equivalent delay-free model, while preserving the optimal solution. Finally, reducing communication overhead without sacrificing convergence performance is nontrivial, especially when the system must converge in fixed time. The triggering condition must be designed to avoid Zeno behavior and to ensure that the event-triggered error does not affect the fixed-time convergence guarantee. Therefore, combining equality constraints, input delays, directed communication, and event-triggered updates in the fixed-time optimization framework is technically demanding, and the research on this topic is necessary and meaningful.

Influenced by those discussion, the main content aims to design an event-triggered distributed fixed-time optimization algorithm to address the DOP with equality constraint and time delay in a directed network. The principal contributions of this work can be stated as:

(i) The DOA proposed in this paper employs an event-triggered communication protocol to alleviate communication overhead. Additionally, compared to the original algorithm, which was mainly used to solve the case where the objective function is the quadratic function, the new algorithm is applicable to cases of the convex function.

(ii) The influence of the input time delay is considered, which is more in line with the actual situation. Subsequently, the method of model order reduction is used to solve it, which can transform the time-delay system into an equivalent delay-free model.

(iii) The fixed-time algorithm designed in this paper can estimate the upper bound of convergence time more accurately, enabling the system to reach the optimal value more quickly. The estimation bound mainly relies on λ_2 and λ_n.

The outline of this paper is as follows. Section 2 is devoted to the preliminaries. In Section 3, the fixed-time DOA is introduced and its convergence is rigorously analyzed. Simulation results are reported and discussed in Section 4. The paper ends with concluding remarks in Section 5.

Notations. Let $\mathbb{R}^m$ denote the m-dimensional real space, $\mathbb{R}^{n\times m}$ the space of real $n \times m$ matrices, and $\mathbb{N}$ the set of natural numbers. A^{T} denotes the transpose of matrix A, $\mathbf{1}_n$ is the $n \times 1$ column vector of all ones. The symbol $\otimes$ denotes the Kronecker product, $|\cdot|$ the absolute value, $\|\cdot\|$ the Euclidean norm, ∇f the gradient of f, and $\nabla^2 f$ the Hessian matrix of f. Moreover, $\mathrm{diag}(\cdot)$ denotes a diagonal matrix, A matrix $A \in \mathbb{R}^{n\times n}$ is called positive if $a_{ij} > 0$ for all $1 \leqslant i, j \leqslant n$. Let $z = [z_1, z_2, \dots, z_m]^{\mathrm{T}} \in \mathbb{R}^m$ be a column vector. Then

$$\mathrm{Sig}(z)^\omega = \left[\mathrm{sig}(z_1)^\omega, \mathrm{sig}(z_2)^\omega, \dots, \mathrm{sig}(z_m)^\omega\right]^{\mathrm{T}}, \quad \mathrm{sig}(z_i)^\omega = \mathrm{sign}(z_i)|z_i|^\omega,$$
$$\mathrm{Sign}(z) = \left[\mathrm{sign}(z_1), \mathrm{sig}(z_2), \dots, \mathrm{sign}(z_m)\right]^{\mathrm{T}},$$

and $\mathrm{sign}(\cdot)$ is a sign function.

2 Preliminaries and problem formulation

In this section, the DOP is first introduced. Then the basic agent dynamics are presented. In addition, several useful lemmas that will be used in the subsequent analysis are provided at the end of this section.

2.1 Optimization problem formulation

The DOP with an equality constraint can be described in this subsection, where every network nodes have a local objective function $f_i(x_i)$, which is only known by node i. The global objective function, denoted as $f(x)$, is defined as the sum of all the local objective function, i.e., $f(x) = \sum_{i=1}^n f_i(x_i)$. The DOP is defined as follows:

$$\min f(x) = \sum_{i=1}^n f_i(x_i) \tag{1}$$

s.t.

$$\sum_{i=1}^n x_i(t) = \sum_{i=1}^n x_i(0) = \xi, \tag{2}$$

where $x_i \in \mathbb{R}^m$, ξ is the $m \times 1$ column vector. The objective is to minimize the sum function $f(x)$ and satisfy the equality constraint (2).

Remark 1. In fact, the DOP (1)–(2) arises from a set of operational constraints. Taking economic dispatch in the smart grid as an example, the variable $x_i(t)$ is defined as the active power output, $f_i(x_i(t))$ denotes the local cost function, ξ denotes the total demand, and $f(x(t))$ denotes the total cost function. The primary objective is to minimize the overall cost subject to the constraint that all demand must be met.

2.2 Agent dynamics

Consider the multi-agent system with n agents. The communication topology is represented by a directed graph $\mathcal{G} = \{\mathcal{V}, \mathcal{E}, A\}$, where $\mathcal{V} = \{v_1, v_2, \dots, v_n\}$ denotes the set of agents, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of communication links, and $A = [a_{ij}]$ denotes the adjacency matrix. If $(v_i, v_j) \in \mathcal{E}$, agent j is a neighbor of agent i with $a_{ij} = 1$; otherwise, $a_{ij} = 0$. It is assumed that there are no self-loops, i.e., $a_{ii} = 0$. The graph is termed strongly connected when there is a path between each pair of distinct nodes. It is weight-balanced if $\sum_{j=1}^n a_{ij} = \sum_{j=1}^n a_{ji}$ for all $i = 1, 2, \dots, n$.

The Laplacian matrix $L \triangleq [l_{ij}] \in \mathbb{R}^{n\times n}$ of graph $\mathcal{G}$ is defined by $l_{ij} = -a_{ij}$ when $i \neq j$ and $l_{ii} = \sum_{j=1}^n a_{ij}$. For a strongly connected and weight-balanced graph, it holds that $L\mathbf{1}_n = 0$ and $\mathbf{1}_n^{\mathrm{T}} L = 0$, and the smallest positive eigenvalue of $L + L^{\mathrm{T}}$ is denoted by $\tilde{\lambda}_2(L + L^{\mathrm{T}}) > 0$.

Under this communication topology, the continuous-time dynamics of each agent i, considering input time delay, can be described as follows:

$$\dot{x}_i(t) = u_i(t - \tau), \quad i = 1, 2, \dots, n,$$

where $x_i(t) \in \mathbb{R}^m$ corresponds to the state, $u_i(t)$ represents its control input, and $\tau > 0$ denotes the input time delay.

Under the input time delay, the following substitution is made:

$$\Omega_i(t) = x_i(t) + \int\limits_{t-\tau}^{t} u_i(s)\,\mathrm{d}s,$$

so

$$\dot{\Omega}_i(t) = \dot{x}_i(t) + u_i(t) - u_i(t - \tau) = u_i(t).$$

Based on this transformation, the delayed system can be converted into an equivalent delay-free form. Moreover, it can be verified that the original equality constraint remains unchanged. Since $\sum_{i=1}^n x_i(t) = \xi$, $\sum_{i=1}^n \dot{x}_i(t) = \mathbf{0} = \sum_{i=1}^n u_i(t - \tau)$,

$$\begin{aligned}\sum_{i=1}^n \Omega_i(t) &= \sum_{i=1}^n \left[x_i(t) + \int\limits_{t-\tau}^{t} u_i(s)\,\mathrm{d}s\right] = \sum_{i=1}^n x_i(t) + \int\limits_{t-\tau}^{t} \sum_{i=1}^n u_i(s)\,\mathrm{d}s \\ &= \sum_{i=1}^n x_i(t) = \xi.\end{aligned}$$

Now, replacing $x_i(t)$ with $\Omega_i(t)$ transforms the optimization problem into the following form:

$$\min f(\Omega) = \sum_{i=1}^n f_i(\Omega_i), \tag{3}$$

s.t.

$$\sum_{i=1}^n \Omega_i(t) = \sum_{i=1}^n \Omega_i(0) = \xi. \tag{4}$$

Remark 2. The original optimization problem (1) and the transformed optimization problem (3) are equivalent. In fact, as $t \to +\infty$, in problem (1), $x_i(t)$ converges to its optimal value x_i^*; in problem (4), $\Omega_i(t)$ converges to its optimal value Ω_i^*. Additionally, the mapping between them is given by

$$\Omega_i(t) = x_i(t) + \int\limits_{t-\tau}^{t} u_i(s)\,\mathrm{d}s.$$

Thus, when $t \to \infty$, $u_i(t) \to 0$, so $\Omega_i(t) - x_i(t) \to 0$ and

$$x_i^* = \lim_{t\to\infty} x_i(t) = \lim_{t\to\infty} \Omega_i(t) - \lim_{t\to\infty} \bigl(\Omega_i(t) - x_i(t)\bigr) = \Omega_i^*.$$

Therefore, $u_i(t)$ can be designed such that when $\Omega_i(t)$ in the transformed system converges to Ω_i^*, $x_i(t)$ also converges to Ω_i^*. Hence, the optimal solutions of the original and transformed optimization problems are identical. Additionally,

$$\lim_{t\to\infty} \sum_{i=1}^{n} f_i\bigl(x_i(t)\bigr) = \sum_{i=1}^{n} f_i(x_i^*) = \sum_{i=1}^{n} f_i(\Omega_i^*) = \lim_{t\to\infty} \sum_{i=1}^{n} f_i\bigl(\Omega_i(t)\bigr),$$

which shows that the objective function values of the two optimization problems are equivalent.

Eventually, this substitution transforms original optimization problem with the input time delay into delay-free model. Thus, this problem can be transformed into the MAS consisting of the identical agents $\Omega_i(t)$, each agent possesses a local objective function denoted as $f_i(\Omega_i(t))$. The task is to collaboratively optimize $f(\Omega)$, while considering global equality constraint.

Remark 3. Comparing with [27], this paper considers input delays and transforms the delayed system into a delay-free system by using model order reduction, which is more consistent with communication/actuation delays in practical engineering.

Assumption 1. The directed communication topology is strongly connected and weight-balanced, and the matrix LL is generalized positive semidefinite, i.e., $x^{\mathrm{T}}LLx \geqslant 0$ for all $x \in \mathbb{R}^n$.

Assumption 2. The objective functions $f_i(x_i(t))$ are strongly convex, and there exists a constant $\delta > 0$ such that $\|\nabla^2 f_i(x_i(t))\| \geqslant \delta$ for all $x_i(t)$ and for all i.

Remark 4. Assumption 1 is crucial for the subsequent fixed-time convergence analysis. In particular, strong connectivity ensures that every agent can indirectly influence every other agent through directed paths, which is necessary for the global information to propagate across the network and for the fixed-time convergence to the optimum. Moreover, the weight-balanced condition guarantees that the global equality constraint is invariant under the algorithm (5).

2.3 Some useful lemmas

Next, we will introduce several important lemmas, which will offer the necessary theoretical foundation for the subsequent analysis of the optimization problems.

Lemma 1. (*See* [1].) *Let* $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \geqslant 0$. *Then*

$$\sum_{i=1}^{n} \varepsilon_i^a \geqslant \left(\sum_{i=1}^{n} \varepsilon_i\right)^a, \quad 0 < a \leqslant 1,$$

$$\sum_{i=1}^{n} \varepsilon_i^a \geqslant n^{1-a}\left(\sum_{i=1}^{n} \varepsilon_i\right)^a, \quad 1 < a < \infty.$$

Lemma 2. (*See* [2].) *Assume that there exists a continuous radially unbounded function* $V : \mathbb{R}^n \to \mathbb{R}_+ \cup \{0\}$ *such that*:

(i) $V(x) = 0 \Leftrightarrow x = 0$;

(ii) *There exist constants* $\mu, \alpha, \beta, p, q, k > 0$ *with* $pk < 1$ *and* $qk > 1$ *for which*

$$\dot{V}(x) \leqslant -\mu V\big(x(t)\big) - \alpha^k V^{pk}\big(x(t)\big) - \beta^k V^{qk}\big(x(t)\big).$$

Then the origin of the system is globally fixed-time stable, and the estimated time $T(x_0)$ *is set as follows*:

$$T(x_0) \leqslant \hat{T} = \frac{\ln(1 + \frac{\mu}{\alpha^k})}{\mu(1 - pk)} + \frac{\ln(1 + \frac{\mu}{\beta^k})}{\mu(qk - 1)}.$$

Remark 5. Comparing with the estimated time $\tilde{T}$ mentioned in Lemma 1 of [15], a more precise setting time $\hat{T}$ can be get by using the Lemma 2. In fact, selecting the appropriate function $f(\varsigma) = \ln(1 + \varsigma) - \varsigma$, when $\varsigma > 0$, $f(\varsigma) < f(0) = 0$. Let $\varsigma = \mu/\alpha^k$. It is easy to show that $\ln(1 + \mu/\alpha^k) < \mu/\alpha^k$, then

$$\frac{\ln(1 + \frac{\mu}{\alpha^k})}{\mu(1 - pk)} < \frac{\frac{\mu}{\alpha^k}}{\mu(1 - pk)} = \frac{1}{\alpha^k(1 - pk)}.$$

In the similar way,

$$\frac{\ln(1 + \frac{\mu}{\beta^k})}{\mu(qk - 1)} < \frac{1}{\beta^k(qk - 1)}.$$

Based on the above discussion, $\hat{T} \leqslant \tilde{T}$, that is why $\hat{T}$ is more precise than $\tilde{T}$.

Lemma 3. (*See* [2].) *Let* $L \in \mathbb{R}^{n\times n}$ *be a Laplacian matrix,* $\rho \in \mathbb{R}$, *and let* $\Theta = \operatorname{diag}\{\theta_1, \theta_2, \ldots, \theta_n\} \in \mathbb{R}^{n\times n}$ *be a positive diagonal matrix. If* LL *is a generalized positive semidefinite matrix, then*

$$\operatorname{Sig}^{\mathrm{T}}(Lx)^{\rho}\Theta L^{\mathrm{T}}x \geqslant 0,$$

where $\operatorname{Sig}(\cdot)^{\rho}$ *denotes a vector-valued function.*

Remark 6. Lemma 3 plays a pivotal role in this work, particularly in the context of the fixed-time stability analysis under directed graphs. By applying Lemma 2 and noting that $\operatorname{Sig}^{\mathrm{T}}(Lx)^{\rho}\Theta L^{\mathrm{T}}x \geqslant 0$ is related with to the Lyapunov function V, the estimated time $T(x_0)$ is established.

3 Main results

In this section, the DOA employing an event-triggered communication mechanism is developed to address (3) subject to (4). Subsequently, a convergence analysis is performed to demonstrate the effectiveness of the DOA.

3.1 Algorithm design

The following DOA with event-triggered communication scheme is designed:

$$\begin{aligned}
\varGamma_i(t) &= \sum_{j\in N_i} a_{ij}\big(\psi_i(t)-\psi_j(t)\big)\\
\varOmega_i(t) &= \sum_{j\in N_i} a_{ij}\big(\varrho_i(t)-\varrho_j(t)\big)+\varOmega_i(0)\\
\dot\varrho_i(t) &= -\alpha\varGamma_i(t_k)-\beta\,\mathrm{sig}\big(\varGamma_i(t_k)\big)^{\omega}-\gamma\,\mathrm{sign}\big(\varGamma_i(t_k)\big),
\end{aligned}\tag{5}$$

where the variable $\varOmega_i(t)\in\mathbb{R}^m$ corresponds to the state in the new multi-agent system, $\psi_i(t)=\partial f_i(\varOmega_i(t))/\partial\varOmega_i(t)\in\mathbb{R}^m$, α, β, and γ are positive constants, $\omega>1$, t_k denotes the triggering instant, $\psi_i(t)$, $\varrho_i(t)$, and $\varGamma_i(t)$ are auxiliary variables, and $\mathrm{sign}(\cdot)$ is the sign function.

Remark 7. In practical engineering fields, communication among agents is often constrained due to limited network resources. Different from [1] and [2], to address this limitation and improve resource utilization efficiency, this paper adopts an event-triggered communication strategy (5) in the design of the fixed-time distributed optimization algorithm.

Algorithm (5) can be cast into a matrix form as follows:

$$\begin{aligned}
\varOmega(t) &= (L\otimes I_m)\varrho(t)+\varOmega(0)\\
\dot\varrho(t) &= -\alpha(L\otimes I_m)\psi(t_k)-\beta\,\mathrm{Sig}^{\omega}\big((L\otimes I_m)\psi(t_k)\big)\\
&\quad-\gamma\,\mathrm{Sign}\big((L\otimes I_m)\psi(t_k)\big),
\end{aligned}\tag{6}$$

where L is the Laplacian matrix, $\varOmega(t)=[\varOmega_1^{\mathrm T}(t),\varOmega_2^{\mathrm T}(t),\dots,\varOmega_n^{\mathrm T}(t)]^{\mathrm T}\in\mathbb{R}^{mn}$, $\varrho(t)=[\varrho_1^{\mathrm T}(t),\varrho_2^{\mathrm T}(t),\dots,\varrho_n^{\mathrm T}(t)]^{\mathrm T}\in\mathbb{R}^{mn}$, and $\psi(t)=[\psi_1^{\mathrm T}(t),\psi_2^{\mathrm T}(t),\dots,\psi_n^{\mathrm T}(t)]\in\mathbb{R}^{mn}$ are auxiliary vectors, $\mathrm{Sig}^{\omega}(\cdot)$ and $\mathrm{Sign}(\cdot)$ are the vector functions.

Remark 8. Combining (6) with $\dot\varOmega(t)=u(t)$ yields

$$\begin{aligned}
u(t) &= \dot\varOmega(t)\\
&= -\alpha(LL\otimes I_m)\psi(t_k)-\beta(L\otimes I_m)\,\mathrm{Sig}\big((L\otimes I_m)\psi(t_k)\big)^{\omega}\\
&\quad-\gamma(L\otimes I_m)\,\mathrm{Sign}\big((L\otimes I_m)\psi(t_k)\big).
\end{aligned}$$

In addition, $\psi(t) = \nabla f(\Omega(t))$, so

$$u(t) = -\alpha(LL \otimes I_m)\nabla f\big(\Omega(t_k)\big) - \beta(L \otimes I_m)\operatorname{Sig}\big((L \otimes I_m)\nabla f\big(\Omega(t_k)\big)\big)^{\omega} \\ - \gamma(L \otimes I_m)\operatorname{Sign}\big((L \otimes I_m)\nabla f\big(\Omega(t_k)\big)\big),$$

which means that a suitable $u(t)$ is designed to address the optimization problem.

Indeed, the second equation in (6) can be reformulated as

$$\begin{aligned}\dot{\varrho}(t) &= -\alpha(L \otimes I_m)\psi(t_k) - \beta\operatorname{Sig}\big((L \otimes I_m)\psi(t_k)\big)^{\omega} - \gamma\operatorname{Sign}\big((L \otimes I_m)\psi(t_k)\big) \\ &= -\big(e(t) + \alpha(L \otimes I_m)\psi(t) + \beta\operatorname{Sig}\big((L \otimes I_m)\psi(t)\big)^{\omega} \\ &\quad + \gamma\operatorname{Sign}\big((L \otimes I_m)\psi(t)\big)\big),\end{aligned} \tag{7}$$

where

$$e(t) = \alpha(L \otimes I_m)\psi(t_k) + \beta\operatorname{Sig}\big((L \otimes I_m)\psi(t_k)\big)^{\omega} + \gamma\operatorname{Sign}\big((L \otimes I_m)\psi(t_k)\big) \\ - \big(\alpha(L \otimes I_m)\psi(t) + \beta\operatorname{Sig}\big((L \otimes I_m)\psi(t)\big)^{\omega} + \gamma\operatorname{Sign}\big((L \otimes I_m)\psi(t)\big)\big).$$

Define

$$g(t) = \big\|e(t)\big\| - \varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\big\|L_1^{\mathrm{T}}\psi\big\|^{\omega} + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2}.$$

The following event-triggered condition is employed to judge whether an update of the state is required:

$$t_{k+1} = \inf_t\big\{t > t_k \bigm| g(t) \geqslant 0\big\},$$

where $\varepsilon \in (0,1)$ denotes the parameter, which serves as a design variable to adjust the triggering frequency, $L_1 = L \otimes I_m$ and $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ satisfy (11)–(13).

Remark 9. Owing to the strongly connected and weight-balanced graph,

$$(\mathbf{1}_n \otimes I_m)^{\mathrm{T}}(L \otimes I_m) = \mathbf{1}_n^{\mathrm{T}}L \otimes I_m = 0.$$

Because of this,

$$\begin{aligned}\sum_{i=1}^{n}\Omega_i(t) &= \sum_{i=1}^{n}\bigg(\sum_{j\in N_i}a_{ij}\big(\varrho_i(t) - \varrho_j(t)\big) + \Omega_i(0)\bigg) \\ &= (\mathbf{1}_n \otimes I_m)^{\mathrm{T}}(L \otimes I_m)\varrho(t) + \xi = \xi,\end{aligned}$$

which ensures that the global equality constraint is satisfied by algorithm (5) at all time instants and for any $\varrho(t)$. As a result, the equality-constrained optimization problem (3)–(4) is reformulated as the following unconstrained CVOP in terms of $\varrho(t)$:

$$\min f\big(\varrho(t)\big) = \sum_{i=1}^{n}f_i\bigg(\sum_{j\in N_i}a_{ij}\big(\varrho_i(t) - \varrho_j(t)\big) + \Omega_i(0)\bigg). \tag{8}$$

Remark 10. Note that $\psi(t) = \nabla f(\Omega(t))$, then

$$\nabla f\big(\varrho(t)\big) = (L \otimes I_m)^{\mathrm{T}} \nabla f\big(\Omega(t)\big) = (L \otimes I_m)^{\mathrm{T}} \psi(t)$$

and

$$\nabla^2 f\big(\varrho(t)\big) = (L \otimes I_m)^{\mathrm{T}} \nabla^2 f\big(\Omega(t)\big)(L \otimes I_m).$$

From (8) the optimal value ϱ^* will be obtained when $\nabla f(\varrho^*) = 0$. In other words, if $\psi(t)$ converges to $\psi^*(\mathbf{1}_n \otimes I_m)$, where $\psi^* \in \mathbb{R}$ is a constant, then $\nabla f(\varrho)$ converges to $\psi^*(L \otimes I_m)^{\mathrm{T}}(\mathbf{1}_n \otimes I_m) = \psi^*(L^{\mathrm{T}}\mathbf{1}_n \otimes I_m) = 0$. Hence, $\psi_i(t) = \psi_j(t)$, and the optimal value can be obtained.

3.2 Convergence analysis

Theorem 1. *Under Assumptions* 1 *and* 2, *the DOA* (5) *is capable of solving the equality-constrained optimization problem* (3)–(4) *within a fixed time, and the corresponding convergence time bound is established*:

$$T(\Omega) \leqslant T_{\max} = \frac{4\ln\big(1 + \frac{\alpha\varepsilon_0\zeta^{1/2}}{2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}}\big)}{\alpha\varepsilon_0\zeta} + \frac{4\ln\big(1 + \frac{\alpha\varepsilon_0\zeta^{(1-\omega)/2}}{2\beta(1-\varepsilon)\varepsilon_1 n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}}\big)}{\alpha\varepsilon_0\zeta(\omega-1)}, \tag{9}$$

where $\zeta = \delta\lambda_2^2/(2\lambda_n)$, α, β, γ, *and* ω *are adjusting parameters in algorithm* (5), $\delta > 0$, $\varepsilon \in (0,1)$ *denotes a triggering parameter, which serves as a design variable to adjust the triggering frequency.* $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$ *satisfy*

$$LL + L^{\mathrm{T}}L^{\mathrm{T}} \geqslant \varepsilon_0 LL^{\mathrm{T}}, \tag{10}$$

$$\beta\psi^{\mathrm{T}}(t)L_1 \operatorname{Sig}\big(L_1\psi(t)\big)^{\omega} \geqslant \varepsilon_1\beta\psi^{\mathrm{T}}(t)L_1^{\mathrm{T}} \operatorname{Sig}\big(L_1\psi(t)\big)^{\omega}, \tag{11}$$

$$\gamma\psi^{\mathrm{T}}(t)L_1 \operatorname{Sign}\big(L_1\psi(t)\big) \geqslant \varepsilon_2\gamma\psi^{\mathrm{T}}(t)L_1^{\mathrm{T}} \operatorname{Sign}\big(L_1\psi(t)\big), \tag{12}$$

$$L^{\mathrm{T}}L \geqslant \varepsilon_3 LL^{\mathrm{T}}. \tag{13}$$

Here $\psi_i(t) = \partial f_i(\Omega_i(t))/\partial\Omega_i(t) \in \mathbb{R}^m$, $\psi(t) = [\psi_1^{\mathrm{T}}(t), \dots, \psi_n^{\mathrm{T}}(t)]^{\mathrm{T}} \in \mathbb{R}^{mn}$, $L_1 = L \otimes I_m$, λ_2 *and* λ_n *are the smallest and largest positive eigenvalues of matrix* $L_1^{\mathrm{T}}L_1$.

Proof. Choose a candidate Lyapunov function as follows:

$$V(t) = \big(f\big(\varrho(t)\big) - f(\varrho^*)\big)^2, \tag{14}$$

where ϱ^* denotes the optimal value, i.e., $f(\varrho^*) \leqslant f(\varrho(t))$.

According to (7),

$$\begin{aligned}\dot V(t) &= 2\big(f\big(\varrho(t)\big) - f(\varrho^*)\big)\nabla^{\mathrm{T}} f\big(\varrho(t)\big)\dot\varrho(t)\\ &= -2\big(f\big(\varrho(t)\big) - f(\varrho^*)\big)\big[\psi^{\mathrm{T}}(t)L_1e(t) + \alpha\psi^{\mathrm{T}}(t)L_1L_1\psi(t)\\ &\quad + \beta\psi^{\mathrm{T}}(t)L_1 \operatorname{Sig}\big(L_1\psi(t)\big)^{\omega} + \gamma\psi^{\mathrm{T}}(t)L_1 \operatorname{Sign}\big(L_1\psi(t)\big)\big].\end{aligned}$$

According to Assumption 1 and $L_1 = L \otimes I_m$, $L_1 L_1$ is a generalized positive semidefinite matrix, thus $\psi^{\mathrm{T}} L_1 L_1 \psi \geqslant 0$. Because of this, $L_1 L_1^{\mathrm{T}}$ and $L_1 L_1 + L_1^{\mathrm{T}} L_1^{\mathrm{T}}$ are positive semidefinite matrices, they have the same zero subspace. So there must exist an $\varepsilon_0 > 0$ satisfying (10). Therefore,

$$\psi^{\mathrm{T}}(t) L_1 L_1 \psi(t) \geqslant \frac{\varepsilon_0}{2} \psi^{\mathrm{T}}(t) L_1 L_1^{\mathrm{T}} \psi(t) = \frac{\varepsilon_0}{2} \bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2.$$

Let

$$\begin{aligned} M_\beta &= \beta \psi^{\mathrm{T}}(t) L_1 \operatorname{Sig}\bigl(L_1 \psi(t)\bigr)^\omega, \qquad & \tilde{M}_\beta &= \beta \psi^{\mathrm{T}}(t) L_1^{\mathrm{T}} \operatorname{Sig}\bigl(L_1 \psi(t)\bigr)^\omega, \\ M_\gamma &= \gamma \psi^{\mathrm{T}}(t) L_1 \operatorname{Sign}\bigl(L_1 \psi(t)\bigr), \qquad & \tilde{M}_\gamma &= \gamma \psi^{\mathrm{T}}(t) L_1^{\mathrm{T}} \operatorname{Sign}\bigl(L_1 \psi(t)\bigr). \end{aligned}$$

According to Lemma 3, $M_\beta, \tilde{M}_\beta, M_\gamma, \tilde{M}_\gamma \geqslant 0$. Combining those above inequalities,

$$\dot{V}(t) \leqslant -2\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \biggl(\psi^{\mathrm{T}}(t) L_1 e(t) + \frac{\alpha \varepsilon_0}{2} \bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2 + M_\beta + M_\gamma \biggr).$$

In fact, when the system reaches the optimal solution, all $\psi_i(t)$ become identical, so $L_1 \psi(t) = 0$. This forces both $\operatorname{Sig}(L_1 \psi(t))^\omega$ and $\operatorname{Sign}(L_1 \psi(t))$ converge to 0, that is to say, $M_\beta, \tilde{M}_\beta, M_\gamma, \tilde{M}_\gamma \to 0$. Hence, these terms vanish at the optimum and decay to zero during the fixed-time convergence process. Suppose that for all $\bar{\varepsilon} > 0$ s.t. $\bar{\varepsilon} \tilde{M}_\beta \geqslant M_\beta$, there must be $M_\beta < 0$, which contradicts to $M_\beta \geqslant 0$. Therefore, there must exist an $\varepsilon_1 > 0$ that satisfies (11). In the same way, there must exist an $\varepsilon_2 > 0$ that satisfies (12). Combining the above discussion, we obtain

$$\begin{aligned} \dot{V}(t) &\leqslant -\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \bigl(2\psi^{\mathrm{T}}(t) L_1 e(t) + \alpha \varepsilon_0 \bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2 + 2\varepsilon_1 \tilde{M}_\beta + 2\varepsilon_2 \tilde{M}_\gamma \bigr) \\ &\leqslant 2\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\| \bigl\| e(t) \bigr\| - \alpha \varepsilon_0 \bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2 \\ &\quad - 2\varepsilon_1 \bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \tilde{M}_\beta - 2\varepsilon_2 \bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \tilde{M}_\gamma. \end{aligned} \tag{15}$$

Let $\Phi(t) = L_1 \psi(t)$, where $\Phi(t) = [\Phi_1^{\mathrm{T}}(t), \Phi_2^{\mathrm{T}}(t), \dots, \Phi_n^{\mathrm{T}}(t)]^{\mathrm{T}}$, $\Phi_i(t) \in \mathbb{R}^m$, so $\Phi(t) \in \mathbb{R}^{mn}$. Obviously, due to Lemma 1, one can get that

$$\begin{aligned} \tilde{M}_\beta &\geqslant \beta n^{(1-\omega)/2} \bigl(\Phi^{\mathrm{T}}(t) \Phi(t)\bigr)^{(1+\omega)/2}, \\ \tilde{M}_\gamma &\geqslant \gamma \bigl(\Phi^{\mathrm{T}}(t) \Phi(t)\bigr)^{1/2}, \end{aligned}$$

where $\omega > 1$. Similarly, there exists $\varepsilon_3 > 0$ satisfying (13), thus

$$\Phi^{\mathrm{T}}(t) \Phi(t) = \psi^{\mathrm{T}}(t) L_1^{\mathrm{T}} L_1 \psi(t) \geqslant \psi^{\mathrm{T}}(t) L_1 L_1^{\mathrm{T}} \psi(t) = \varepsilon_3 \| \nabla f(\varrho(t)) \|^2.$$

Therefore,

$$\begin{aligned} \tilde{M}_\beta &\geqslant \beta n^{(1-\omega)/2} \varepsilon_3^{(1+\omega)/2} \bigl(\bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2 \bigr)^{(1+\omega)/2}, \\ \tilde{M}_\gamma &\geqslant \gamma \varepsilon_3^{1/2} \bigl(\bigl\| \nabla f\bigl(\varrho(t)\bigr) \bigr\|^2 \bigr)^{1/2}. \end{aligned}$$

On the other hand, due to

$$\bigl\| e(t) \bigr\| \leqslant \varepsilon \varepsilon_1 \beta n^{(1-\omega)/2} \varepsilon_3^{(1+\omega)/2} \bigl\| L_1^{\mathrm{T}} \psi \bigr\|^\omega + \gamma \varepsilon \varepsilon_2 \varepsilon_3^{1/2},$$

we have

$$\begin{aligned}\|\nabla f(\varrho(t))\|\|e(t)\| &\leqslant \varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\|L_1^{\mathrm{T}}\psi\|^{\omega+1} + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2}\|L_1^{\mathrm{T}}\psi\| \\ &\leqslant \varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\bigl(\|\nabla f(\varrho(t))\|^2\bigr)^{(\omega+1)/2} \\ &\quad + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2}\bigl(\|\nabla f(\varrho(t))\|^2\bigr)^{1/2}.\end{aligned}$$

Substituting these inequalities back into (15) yields

$$\begin{aligned}\dot V(t) \leqslant{}& -\bigl(f(\varrho(t)) - f(\varrho^*)\bigr)\bigl(\alpha\varepsilon_0\|\nabla f(\varrho(t))\|^2 \\ &+ 2\beta(1-\varepsilon)\varepsilon_1 n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\bigl(\|\nabla f(\varrho(t))\|^2\bigr)^{(\omega+1)/2} \\ &+ 2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}\bigl(\|\nabla f(\varrho(t))\|^2\bigr)^{1/2}\bigr).\end{aligned} \tag{16}$$

In addition, according to Assumption 2,

$$\begin{aligned}f(\varrho(t)) - f(\varrho^*) &= \nabla^{\mathrm{T}} f(\varrho^*)\bigl(\varrho(t)-\varrho^*\bigr) + \frac{1}{2}\bigl(\varrho(t)-\varrho^*\bigr)^{\mathrm{T}}\nabla^2 f(\hat\varrho)\bigl(\varrho(t)-\varrho^*\bigr) \\ &\geqslant \frac{\delta}{2}\bigl(\varrho(t)-\varrho^*\bigr)^{\mathrm{T}} L_1^{\mathrm{T}} L_1\bigl(\varrho(t)-\varrho^*\bigr) \\ &\geqslant \frac{\delta}{2\lambda_n}\bigl(L_1^{\mathrm{T}} L_1(\varrho(t)-\varrho^*)\bigr)^{\mathrm{T}} L_1^{\mathrm{T}} L_1\bigl(\varrho(t)-\varrho^*\bigr),\end{aligned}$$

where $\delta > 0$. The eigenvalues of $L^{\mathrm{T}}L$ are $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$. The corresponding orthonormal eigenvectors are $\mathbf{1}_n, \upsilon_2, \ldots, \upsilon_n$ satisfying $\|\upsilon_i\| = 1$ and $\upsilon_i^{\mathrm{T}}\upsilon_j = 0$ for $i \neq j$. Since $L_1 = L \otimes I_m$, it follows that

$$L_1^{\mathrm{T}} L_1 = (L_1 \otimes I_m)^{\mathrm{T}}(L_1 \otimes L_1) = \bigl(L^{\mathrm{T}} \otimes I_m\bigr)(L \otimes I_m) = \bigl(L^{\mathrm{T}} L\bigr) \otimes I_m$$

Therefore, the eigenvalues of $L_1^{\mathrm{T}} L_1$ are $0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ with each λ_i having multiplicity m. Consequently, the corresponding orthonormal eigenvectors are $\upsilon_i \otimes e_k$, where $\{e_k\}_{k=1}^m$ forms the orthonormal basis for $\mathbb{R}^m$. Moreover,

$$(\upsilon_i \otimes e_k)^{\mathrm{T}}(\upsilon_j \otimes e_{k'}) = \bigl(\upsilon_i^{\mathrm{T}}\upsilon_j\bigr) \otimes \bigl(e_k^{\mathrm{T}} e_{k'}\bigr) = 0.$$

Hence, the vector can be expressed as

$$\varrho(t) - \varrho^* = \sum_{i=1}^{n}\sum_{k=1}^{m}\kappa_{i,k}(\upsilon_i \otimes e_k),$$

where $\kappa_{i,k}$ $(i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m)$ are constants.

Let

$$\upsilon = \sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}(\upsilon_i \otimes e_k), \quad \|\upsilon\|^2 = \sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}^2,$$
$$L_1^{\mathrm{T}} L_1(\varrho(t)-\varrho^*) = \bigl(\bigl(L^{\mathrm{T}}L\bigr)\otimes I_m\bigr)\bigl(\varrho(t)-\varrho^*\bigr),$$

and

$$\bigl(\bigl(L^{\mathrm{T}}L\bigr)\otimes I_m\bigr)(\upsilon_i \otimes e_k) = \bigl(L^{\mathrm{T}}L\upsilon_i\bigr)\otimes e_k = \lambda_i\upsilon_i \otimes e_k = \lambda_i(\upsilon_i \otimes e_k),$$

that is

$$\big((L^{\mathrm{T}}L)\otimes I_m\big)\big(\varrho(t)-\varrho^*\big)=\sum_{i=1}^{n}\sum_{k=1}^{m}\kappa_{i,k}\lambda_i(\upsilon_i\otimes e_k).$$

Since $\lambda_1=0$, based on the above discussion,

$$\begin{aligned}
&\big(L_1^{\mathrm{T}}L_1\big(\varrho(t)-\varrho^*\big)\big)^{\mathrm{T}}L_1^{\mathrm{T}}L_1\big(\varrho(t)-\varrho^*\big)\\
&\quad=\Bigg(\sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}\lambda_i(\upsilon_i\otimes e_k)\Bigg)^{\mathrm{T}}\sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}\lambda_i(\upsilon_i\otimes e_k)\\
&\quad=\sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}^2\lambda_i^2\geqslant\lambda_2^2\sum_{i=2}^{n}\sum_{k=1}^{m}\kappa_{i,k}^2\geqslant\lambda_2^2\|\upsilon\|^2.
\end{aligned}$$

Then it can be obtained that

$$f\big(\varrho(t)\big)-f(\varrho^*)\geqslant\frac{\delta\lambda_2^2}{2\lambda_n}\|\upsilon\|^2.$$

On the other hand,

$$f(\varrho^*)-f\big(\varrho(t)\big)=\nabla^{\mathrm{T}}f\big(\varrho(t)\big)\big(\varrho^*-\varrho(t)\big)+\frac{1}{2}\big(\varrho^*-\varrho(t)\big)^{\mathrm{T}}\nabla^2f(\hat{\varrho})\big(\varrho^*-\varrho(t)\big),$$

so

$$\begin{aligned}
f\big(\varrho(t)\big)-f(\varrho^*)&=\nabla^{\mathrm{T}}f\big(\varrho(t)\big)\big(\varrho(t)-\varrho^*\big)-\frac{1}{2}\big(\varrho^*-\varrho(t)\big)^{\mathrm{T}}\nabla^2f(\hat{\varrho})\big(\varrho^*-\varrho(t)\big)\\
&\leqslant\nabla^{\mathrm{T}}f\big(\varrho(t)\big)\big(\varrho(t)-\varrho^*\big)=\psi^{\mathrm{T}}L_1\big(\kappa_1\mathbf{1}_n\otimes I_m+\upsilon\big)\\
&=\kappa_1\psi^{\mathrm{T}}L_1(\mathbf{1}_n\otimes I_m)+\nabla^{\mathrm{T}}f\big(\varrho(t)\big)\upsilon\leqslant\big\|\nabla f\big(\varrho(t)\big)\big\|\|\upsilon\|.
\end{aligned}$$

Combining those inequalities,

$$\big\|\nabla f\big(\varrho(t)\big)\big\|^2\geqslant\frac{\delta\lambda_2^2}{2\lambda_n}\big(f\big(\varrho(t)\big)-f(\varrho^*)\big).\tag{17}$$

Substituting (17) back into (16),

$$\begin{aligned}
\dot{V}(t)&\leqslant-\big(f\big(\varrho(t)\big)-f(\varrho^*)\big)\Bigg(\alpha\varepsilon_0\frac{\delta\lambda_2^2}{2\lambda_n}\big(f\big(\varrho(t)\big)-f(\varrho^*)\big)\\
&\qquad+2\beta(1-\varepsilon)\varepsilon_1n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\bigg(\frac{\delta\lambda_2^2}{2\lambda_n}\big(f\big(\varrho(t)\big)-f(\varrho^*)\big)\bigg)^{(\omega+1)/2}\\
&\qquad+2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}\bigg(\frac{\delta\lambda_2^2}{2\lambda_n}\big(f\big(\varrho(t)\big)-f(\varrho^*)\big)\bigg)^{1/2}\Bigg)\\
&=-\alpha\varepsilon_0\frac{\delta\lambda_2^2}{2\lambda_n}V(t)-2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}\bigg(\frac{\delta\lambda_2^2}{2\lambda_n}\bigg)^{1/2}V^{3/4}(t)\\
&\qquad-2\beta(1-\varepsilon)\varepsilon_1n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\bigg(\frac{\delta\lambda_2^2}{2\lambda_n}\bigg)^{(\omega+1)/2}V^{(\omega+3)/4}(t)\\
&=-aV(t)-b^kV^{pk}(t)-c^kV^{qk}(t),
\end{aligned}$$

where

$$a = \alpha\varepsilon_0 \frac{\delta\lambda_2^2}{2\lambda_n}, \qquad b = 2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}\left(\frac{\delta\lambda_2^2}{2\lambda_n}\right)^{1/2},$$
$$c = 2\beta(1-\varepsilon)\varepsilon_1 n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\left(\frac{\delta\lambda_2^2}{2\lambda_n}\right)^{(\omega+1)/2},$$
$$k = 1, \qquad p = \frac{3}{4} \in (0,1), \qquad q = \frac{3+\omega}{4} > 1.$$

With the aid of Lemma 2, it is demonstrated that $V(t)$ converges to 0 with the convergence time bound given by (9).

With the aid of the function (14) and the estimated time bound (9) for all $t \geqslant T(\Omega)$, $\varrho(t) = \varrho^*$, which implies that the CVOP (3) subject to equality constraint (4) is addressed within a fixed time, and $\Omega_i(t)$ can reach the optimization value Ω^*. □

Theorem 2. *The event-triggered communication scheme presented in this work is capable of precluding Zeno behavior, ensuring that no accumulation of triggering events occurs in fixed time.*

Proof. First, $\Gamma_i(t) = [\Gamma_{i1}(t), \Gamma_{i2}(t), \dots, \Gamma_{im}(t)]^{\mathrm{T}} \in \mathbb{R}^m$, so

$$\operatorname{sig}(\Gamma_i(t))^\omega = \begin{bmatrix} \operatorname{sign}(\Gamma_{i1}(t))|\Gamma_{i1}(t)|^\omega \\ \operatorname{sign}(\Gamma_{i2}(t))|\Gamma_{i2}(t)|^\omega \\ \vdots \\ \operatorname{sign}(\Gamma_{im}(t))|\Gamma_{im}(t)|^\omega \end{bmatrix} = \begin{bmatrix} \Gamma_{i1}(t)|\Gamma_{i1}(t)|^{\omega-1} \\ \Gamma_{i2}(t)|\Gamma_{i2}(t)|^{\omega-1} \\ \vdots \\ \Gamma_{im}(t)|\Gamma_{im}(t)|^{\omega-1} \end{bmatrix}.$$

Let $Q(p) = p|p|^{\omega-1}$, it can be proved that

$$\dot{Q}(p) = \omega|p|^{\omega-1}.$$

In addition,

$$e(t) = \alpha L_1\psi(t_k) + \beta \operatorname{Sig}\bigl(L_1\psi(t_k)\bigr)^\omega + \gamma \operatorname{Sign}\bigl(L_1\psi(t_k)\bigr)$$
$$- \bigl(\alpha L_1\psi(t) + \beta \operatorname{Sig}\bigl(L_1\psi(t)\bigr)^\omega + \gamma \operatorname{Sign}\bigl(L_1\psi(t)\bigr)\bigr),$$

$L_1 = L \otimes I_m$, $\Gamma(t) = L_1\psi(t)$, so

$$\dot{e}(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\alpha\Gamma(t) + \beta \operatorname{Sig}\bigl(\Gamma(t)\bigr)^\omega + \gamma \operatorname{Sign}\bigl(\Gamma(t)\bigr)\bigr).$$

Since $\mathrm{d}\operatorname{Sig}(\Gamma(t))^\omega/\mathrm{d}t = (\partial \operatorname{Sig}(\Gamma)^\omega/\partial\Gamma)\dot{\Gamma}(t)$, let

$$J(\Gamma) = \frac{\partial \operatorname{Sig}(\Gamma)^\omega}{\partial\Gamma} = \operatorname{diag}\bigl(J_1(\Gamma_1), J_2(\Gamma_2), \dots, J_n(\Gamma_n)\bigr) \in \mathbb{R}^{mn}.$$

where each $J_i(\Gamma_i(t))$ is a diagonal matrix, $J_i(\Gamma_i) = \omega \operatorname{diag}(|\Gamma_{i1}(t)|^{\omega-1}, |\Gamma_{i2}(t)|^{\omega-1}, \dots, |\Gamma_{im}(t)|^{\omega-1})$, and $J_i(\Gamma_i(t)) \leqslant \omega \max_{j=1,2,\dots,m}\{|\Gamma_{ij}(t)|^{\omega-1}\}I_m$, so

$$J(\Gamma) \leqslant \omega \max_{i=1,2,\dots,n;\, j=1,2,\dots,m}\bigl\{\bigl|\Gamma_{ij}(t)\bigr|^{\omega-1}\bigr\}I_{mn}.$$

Based on the above discussion,

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Sig}\big(\varGamma(t)\big)^{\omega} = J(\varGamma(t))\dot{\varGamma}(t),$$

so

$$\dot{e}(t) = -\alpha\dot{\varGamma}(t) - \beta J\big(\varGamma(t)\big)\dot{\varGamma}(t).$$

Then, for $t \in [t_k, t_{k+1}]$, $k \in \mathbb{N}$, it follows that

$$\begin{aligned} D^{+}\big\|e(t)\big\| &\leqslant \big\|\dot{e}(t)\big\| = \big\|-\alpha\dot{\varGamma}(t) - \beta J\big(\varGamma(t)\big)\dot{\varGamma}(t)\big\| \\ &\leqslant \big\|\alpha I_{mn} + \beta J\big(\varGamma(t)\big)\big\|\big\|\dot{\varGamma}(t)\big\|. \end{aligned}$$

Since $\varGamma(t) = L_1\nabla f(\varOmega(t))$ and $\dot{\varGamma}(t) = L_1\nabla^2 f(\varOmega(t))L_1\dot{\varrho}$, it follows from (5) that both $\varGamma(t)$ and $\dot{\varGamma}(t)$ are bounded for $t \in [t_k, t_{k+1}]$.

Denote

$$\varDelta(t) = \max_{i=1,2,\dots,n;\, j=1,2,\dots,m}\big\{\big|\varGamma_{ij}(t)\big|^{\omega-1}\big\}$$

and

$$\varLambda(t) = \max_{t_k \leqslant s \leqslant t}\big\{\big\|\big(\alpha + \beta\omega\varDelta(s)\big)I_{mn}\big\|\big\|\dot{\varGamma}(s)\big\|\big\}.$$

Because $e(t_k) = 0$, one can get

$$\big\|e(t)\big\| \leqslant \int_{t_k}^{t}\big\|\dot{e}(s)\big\|\,\mathrm{d}s \leqslant \int_{t_k}^{t}\big\|\alpha I_{mn} + \beta J\big(\varGamma(s)\big)\big\|\big\|\dot{\varGamma}(s)\big\|\,\mathrm{d}s \leqslant \varLambda(t)(t - t_k).$$

As we know, the next event occurs upon the condition $g(t) = 0$ or $||e(t)|| = \varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\|\varGamma(t)\|^{\omega} + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2}$. Therefore,

$$\begin{aligned} \big\|e(t_{k+1})\big\| &= \varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\big\|\varGamma(t_{k+1})\big\|^{\omega} + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2} \\ &\leqslant \varLambda(t_{k+1})(t_{k+1} - t_k). \end{aligned} \tag{18}$$

Form (18) it can be obtained that

$$t_{k+1} - t_k \geqslant \frac{\varepsilon\varepsilon_1\beta n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}\|\varGamma(t_{k+1})\|^{\omega} + \gamma\varepsilon\varepsilon_2\varepsilon_3^{1/2}}{\varLambda(t_{k+1})} > 0.$$

Thus, the Zeno behavior does not occur. □

According to Lemma 2,

$$\lim_{t\to T(\varOmega)} V(t) = 0,$$

and

$$\begin{aligned} T(\varOmega) &\leqslant T_{\max} \\ &= \frac{4\ln(1 + \frac{\alpha\varepsilon_0\zeta^{1/2}}{2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}})}{\alpha\varepsilon_0\zeta} + \frac{4\ln(1 + \frac{\alpha\varepsilon_0\zeta^{(1-\omega)/2}}{2\beta(1-\varepsilon)\varepsilon_1 n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}})}{\alpha\varepsilon_0\zeta(\omega-1)}, \end{aligned}$$

where $\zeta = \delta\lambda_2^2/(2\lambda_n)$, which indicates that $u_i(t)$ converges to zero at $t = T(\Omega) \geqslant T_{\max}$, and $\int_{t-\tau}^{t} u_i(s)\,\mathrm{d}s$ converges to zero when $t \geqslant T_{\max} + \tau$. Thus

$$x_i(t) - x_j(t) = \left(\Omega_i(t) - \int\limits_{t-\tau}^{t} u_i(s)\,\mathrm{d}s\right) - \left(\Omega_j(t) - \int\limits_{t-\tau}^{t} u_j(s)\,\mathrm{d}s\right)$$
$$= \Omega^* - \Omega^* = 0.$$

In other words, $x_i(t)$ can reach the optimization value $x^* = \Omega^*$ when $t = T(x) \geqslant T_{\max} + \tau$. Hence, fixed-time convergence to both the solution of the constrained optimization problem (1)–(2) and consensus is guaranteed, with the upper bound on $T(x)$ provided by

$$T(x) \leqslant \frac{4\ln(1 + \frac{\alpha\varepsilon_0\zeta^{1/2}}{2\gamma(1-\varepsilon)\varepsilon_2\varepsilon_3^{1/2}})}{\alpha\varepsilon_0\zeta} + \frac{4\ln(1 + \frac{\alpha\varepsilon_0\zeta^{(1-\omega)/2}}{2\beta(1-\varepsilon)\varepsilon_1 n^{(1-\omega)/2}\varepsilon_3^{(1+\omega)/2}})}{\alpha\varepsilon_0\zeta(\omega - 1)} + \tau, \tag{19}$$

where $\zeta = \delta\lambda_2^2/(2\lambda_n)$.

Remark 11. Owing to the fact that under the directed graph, Laplacian matrix is not symmetric, the proof of Theorem 1 necessitates the use of several inequalities, which leads to the emergence of several constant parameters, such as ε_0, ε_1, ε_2, ε_3. Since those parameters are strongly associated with the Laplacian matrix L, smaller values of them will cause an increase in the time estimate. On the other hand, under an undirected and connected communication topology, those parameters can be entirely dispensed with, and the time estimate is rendered more accurate as a result.

Corollary 1. *Under an undirected and connected communication topology, the equality constrained optimization problem* (3)–(4) *is guaranteed to be solved in fixed time by the DOA* (5)*, and the estimated time is obtained as*

$$T(x) \leqslant \frac{4\ln(1 + \frac{\alpha\hat{\zeta}^{1/2}}{2\gamma(1-\varepsilon)})}{\alpha\hat{\zeta}} + \frac{4\ln(1 + \frac{\alpha\hat{\zeta}^{(1-\omega)/2}}{2\beta(1-\varepsilon)n^{(1-\omega)/2}})}{\alpha\hat{\zeta}(\omega - 1)}, \tag{20}$$

where $\hat{\zeta} = \delta\hat{\lambda}_2^2/2$, α, β, γ, *and* ω *are adjusting parameters in algorithm* (5)*,* $\delta > 0$, $\varepsilon \in (0, 1)$ *serves as the parameter that can be tuned to adjust the event triggering frequency,* $\hat{\lambda}_2$ *denotes the smallest positive eigenvalue of the Laplacian matrix* L*.*

Proof. Since the Laplacian matrix is symmetric and the Lyapunov function is defined by (14), it follows that

$$\begin{aligned}\dot{V}(t) \leqslant{}& -\big(f\big(\varrho(t)\big) - f(\varrho^*)\big)\\ &\times \big(2\alpha\big\|\nabla f\big(\varrho(t)\big)\big\|^2 + 2\beta(1-\varepsilon)n^{(1-\omega)/2}\big(\big\|\nabla f\big(\varrho(t)\big)\big\|^2\big)^{(\omega+1)/2}\\ &+ 2\gamma(1-\varepsilon)\big(\big\|\nabla f\big(\varrho(t)\big)\big\|^2\big)^{1/2}\big).\end{aligned} \tag{21}$$

According to Theorem 1,

$$f\bigl(\varrho(t)\bigr) - f(\varrho^*) \geqslant \frac{\delta\hat{\lambda}_2^2}{2}\|\upsilon\|^2, \tag{22}$$

where $\hat{\lambda}_2$ is the smallest positive eigenvalue of the Laplacian matrix L.

Substituting (22) into (21),

$$\begin{aligned}
\dot{V}(t) \leqslant{} & -\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr)\biggl(\alpha\delta\hat{\lambda}_2^2\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr) \\
& + 2\beta(1-\varepsilon)n^{(1-\omega)/2}\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr)\biggr)^{(\omega+1)/2} \\
& + 2\gamma(1-\varepsilon)\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\bigl(f\bigl(\varrho(t)\bigr) - f(\varrho^*)\bigr)\biggr)^{1/2}\biggr) \\
={} & -\alpha\frac{\delta\hat{\lambda}_2^2}{2}V(t) - 2\gamma(1-\varepsilon)\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\biggr)^{1/2}V^{3/4}(t) \\
& - 2\beta(1-\varepsilon)n^{(1-\omega)/2}\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\biggr)^{(\omega+1)/2}V^{(\omega+3)/4}(t) \\
={} & -\tilde{a}V(t) - \tilde{b}^kV^{pk}(t) - \tilde{c}^kV^{qk}(t),
\end{aligned}$$

where

$$\begin{gathered}
\tilde{a} = \alpha\frac{\delta\hat{\lambda}_2^2}{2}, \qquad \tilde{b} = 2\gamma(1-\varepsilon)\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\biggr)^{1/2}, \\
\tilde{c} = 2\beta(1-\varepsilon)n^{(1-\omega)/2}\biggl(\frac{\delta\hat{\lambda}_2^2}{2}\biggr)^{(\omega+1)/2}, \\
k = 1, \qquad p = \frac{3}{4} \in (0,1), \qquad q = \frac{3+\omega}{4} > 1.
\end{gathered}$$

Applying Lemma 2, it is demonstrated that $V(t)$ converges to 0 with the convergence time bound given by (20).

Following the method presented in Theorem 1, the Zeno behavior can also be excluded. □

4 Numerical simulations

A numerical example of economic dispatch is introduced in this section to demonstrate the effectiveness of algorithm (5). In the smart grids, the economic dispatch problem involving six generators is a very typical class of distributed optimization problems. Specifically, the cost coefficients and the network topology are adapted from [2] in our paper, with modifications to the Laplacian matrix to satisfy the strongly connected and weight-balanced condition.The communication graph, which is strongly connected and

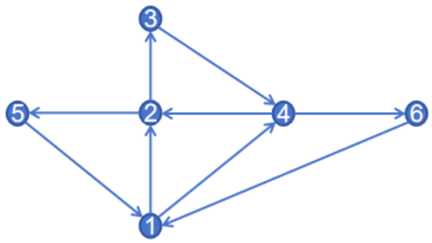

Figure 1. Communication network.

Table 1. The coefficients

Node	1	2	3	4	5	6
α_i	0.096	0.071	0.108	0.073	0.065	0.092
β_i	1.35	2.28	3.21	1.42	4.22	3.51
γ_i	51	42	33	49	56	78

weight-balanced as shown in Fig. 1, constitutes the network topology, and the Laplacian matrix L is presented as follows:

$$L = \begin{bmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ -1 & 2 & 0 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}.$$

Through calculation, it can be obtained that the given L in the simulation satisfies Assumption 1, i.e., $x^{\mathrm{T}}LLx \geqslant 0$. In the economics dispatch problem, $x_i(t) \in \mathbb{R}$ is the active power, ξ is the total demand. $f_i(x_i(t)) = \alpha_i x_i(t)^2 + \beta_i x_i(t) + \gamma_i$ are the cost functions in which α_i, β_i, and γ_i represent the cost coefficient. Thus,

$$\min f(x) = \sum_{i=1}^{6} f_i(x_i) \quad \text{s.t.} \quad \sum_{i=1}^{6} x_i(t) = \sum_{i=1}^{6} x_i(0) = \xi.$$

In the simulation, the cost coefficients α_i, β_i, and γ_i are given in Table 1. We choose the initial states as $x(0) = [70, 100, 80, 70, 90, 90]^{\mathrm{T}}$ MW and $\varrho(0) = [10, 10, 10, 10, 10, 10]^{\mathrm{T}}$, the total demand ξ is assumed to be 500 MW. Let $\varepsilon_1 = 10$, $\varepsilon_2 = 0.1$, $\varepsilon_3 = 0.5$, and let the tuning parameters be chosen as $\alpha = 80$, $\beta = 10$, $\gamma = 10$, and $\delta = 0.1$. Moreover, system parameter $\omega = 1.5$, time delay constant $\tau = 0.05$, and the triggering parameter $\varepsilon = 0.6$.

By calculation, it can be obtained that $\zeta = 0.000318$. Therefore, from (19) and the above parameter settings it follows that $T_{\max} = 0.441905$ s. Moreover, it can be clearly observed from Fig. 2 that the state trajectories of $x_i(t)$ are very close to the optimal value $x^* = [77.645, 98.437, 60.408, 101.627, 92.599, 69.283]^{\mathrm{T}}$ MW. The evolution of the

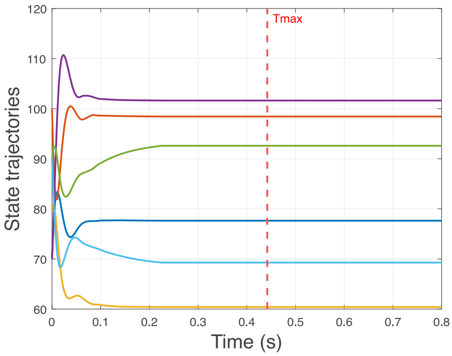

Figure 2. The state trajectories.

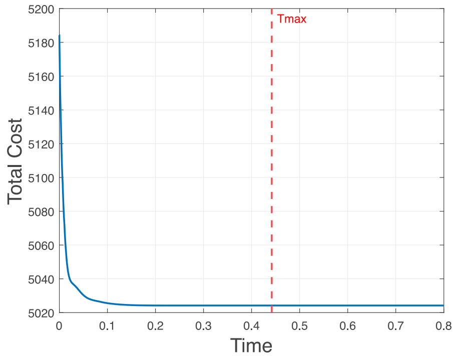

Figure 3. The cost function.

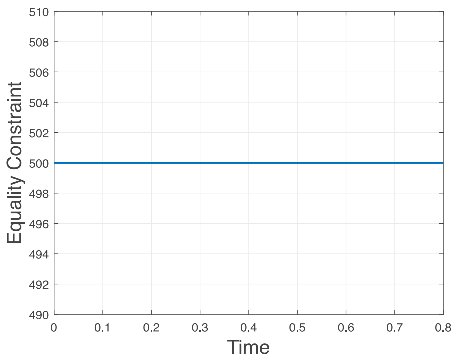

Figure 4. The equality constraint ξ.

objective function and the equality constraint is illustrated in Figs. 3 and 4, respectively. It can be observed that when the time exceeds 0.2 s, the objective function reaches its minimum value, while the equality constraint remains satisfied.

5 Conclusion

This paper addresses the DOP for systems with input time delay and an equality constraint. A distributed protocol is designed by employing model order reduction to transform the original time-delay system into an equivalent delay-free model. Based on Lyapunov theory, the fixed-time convergence of the proposed protocol is established, and an upper bound on the estimated time is derived. To reduce communication overhead, an event-triggered communication scheme is incorporated. Subsequently, a numerical simulation is introduced to demonstrate the effectiveness of the proposed control scheme. Future work will focus on observer-based fixed-time DOP.

Conflicts of interest. The authors declare no conflicts of interest.

References

1. G. Chen, Z. Li, A fixed-time convergent algorithm for distributed convex optimization in multi-agent systems, *Automatica*, **95**:539–543, 2018, `https://doi.org/10.1016/j.automatica.2018.05.032`.
2. H. Dai, J. Jia, L. Yan, X. Fang, W. Chen, Distributed fixed-time optimization in economic dispatch over directed networks, *IEEE Trans. Ind. Inf.*, **17**(5):3011–3019, 2020, `https://doi.org/10.1109/TII.2020.3010282`.
3. M. Gu, Z. Yu, H. Jiang, Distributed constraint optimization for discrete-time multiagent systems with event-triggered communication, *Nonlinear Anal. Model. Control*, **30**(1):119–134, 2025, `https://doi.org/0009-0000-9996-4755`.
4. M. Hong, Z.Q. Luo, On the linear convergence of the alternating direction method of multipliers, *Math. Program.*, **162**(1):165–199, 2017, `https://doi.org/10.1007/s10107-016-1034-2`.
5. Z. Hu, J. Yang, Distributed finite-time optimization for second order continuous-time multiple agents systems with time-varying cost function, *Neurocomputing*, **287**:173–184, 2018, `https://doi.org/10.1016/j.neucom.2018.01.082`.
6. Y. Huang, X. Zeng, J. Sun, Z. Meng, Distributed event-triggered algorithm for convex optimization with coupled constraints, *Automatica*, **170**:111877, 2024, `https://doi.org/10.1016/j.automatica.2024.111877`.
7. J. Liu, D.W.C. Ho, L. Li, A generic algorithm framework for distributed optimization over the time-varying network with communication delays, *IEEE Trans. Autom. Control*, **69**(1):371–378, 2023, `https://doi.org/10.1109/TAC.2023.3264784`.
8. Y. Liu, G. Zhuang, X. Xie, Q. Ma, Impulsive observer-based admissibilization for delayed degenerate jump systems and application to DCM-IP device, *Nonlinear Anal., Hybrid Syst.*, **50**:101395, 2023, `https://doi.org/10.1016/j.nahs.2023.101395`.
9. L. Ma, C. Hu, S. Wen, Z. Yu, H. Jiang, Fixed-time distributed optimization via edge-based adaptive algorithms, *IEEE Trans. Syst. Man Cybern., Syst.*, **55**(5):3436–3448, 2025, `https://doi.org/10.1109/TSMC.2025.3534629`.
10. Z.A. Malik, M. Rehan, W. Ahmed, I. Ahmed, C.K. Ahn, Consensus-based event-triggered distributed optimization for a network of agents with cubic objective functions, *IEEE Trans. Network Sci. Eng.*, **12**(6):4939–4951, 2025, `https://doi.org/10.1109/TNSE.2025.3578140`.
11. Q. Meng, S. Hussain, F. Luo, Z. Wang, X. Jin, An online reinforcement learning-based energy management strategy for microgrids with centralized control, *IEEE Trans. Ind. Appl.*, **61**(1):1501–1510, 2024, `https://doi.org/10.1109/TIA.2024.3430264`.
12. A. Nedic, A. Ozdaglar, Distributed subgradient methods for multi-agent optimization, *IEEE Trans. Autom. Control*, **54**(1):48–61, 2009, `https://doi.org/10.1109/TAC.2008.2009515`.
13. B. Ning, Q.-L. Han, Z. Zuo, Distributed optimization for multiagent systems: An edge-based fixed-time consensus approach, *IEEE Trans. Cybern.*, **49**(1):122–132, 2017, `https://doi.org/10.1109/TCYB.2017.2766762`.

14. Y. Pang, G. Zhuang, J. Xia, X. Xie, Fault detection and robust security control for implicit jump systems against random FDI attacks and packet loss under memorized output-perceptive event-triggered protocol, *IEEE Trans. Inf. Forensics Secur.*, **19**:7362–7373, 2024, `https://doi.org/10.1109/TIFS.2024.3434739`.
15. A. Polyakov, Nonlinear feedback design for fixed-time stabilization of linear control systems, *IEEE Trans. Autom. Control*, **57**(8):2106–2110, 2011, `https://doi.org/10.1109/TAC.2011.2179869`.
16. Y. Song, J. Cao, L. Rutkowski, A fixed-time distributed optimization algorithm based on event-triggered strategy, *IEEE Trans. Network Sci. Eng.*, **9**(3):1154–1162, 2021, `https://doi.org/10.1109/TNSE.2021.3133541`.
17. Y. Song, W. Chen, Finite-time convergent distributed consensus optimisation over networks, *IET Control Theory Appl.*, **10**(11):1314–1318, 2016, `https://doi.org/10.1049/iet-cta.2015.1051`.
18. R. Tang, W. Zhu, H. Pu, Event-triggered distributed optimization of multi-agent systems with time delay, *Math. Biosci. Eng.*, **20**(12):20712–20726, 2023, `https://doi.org/10.3934/mbe.2023916`.
19. A.G. Tsikalakis, N.D. Hatziargyriou, Centralized control for optimizing microgrids operation, in *2011 IEEE Power and Energy Society General Meeting*, IEEE, Piscataway, NJ, 2011, pp. 1–8, `https://doi.org/10.1109/PES.2011.6039737`.
20. J. Wang, N. Elia, A control perspective for centralized and distributed convex optimization, in *2011 50th IEEE Conference on Decision and Control and European Control Conference*, IEEE, Piscataway, NJ, 2011, pp. 3800–3805, `https://doi.org/10.1109/CDC.2011.6161503`.
21. S. Wang, F. Roosta-Khorasani, P. Xu, M.W. Mahoney, GIANT: Globally improved approximate Newton method for distributed optimization, in *Advances in Neural Information Processing Systems 31 (NeurIPS 2018)*, Currant Associates, Red Hook, NY, 2018, pp. 2332–2342.
22. X.F. Wang, Y. Hong, X.M. Sun, K.Z. Liu, Distributed optimization for resource allocation problems under large delays, *IEEE Trans. Ind. Electron.*, **66**(12):9448–9457, 2019, `https://doi.org/10.1109/TIE.2019.2891406`.
23. Z. Wu, C. Zhang, Event triggered distributed fixed-time optimal charging control for electric vehicle, *Optim. Eng.*, **26**:1495–1512, 2025, `https://doi.org/10.1007/s11081-024-09945-w`.
24. X. Xu, Z. Yu, D. Huang, H. Jiang, Distributed optimization for multi-agent systems with communication delays and external disturbances under a directed network, *Nonlinear Anal. Model. Control*, **28**(3):412–430, 2023, `https://doi.org/10.15388/namc.2023.28.31563`.
25. Y. Xu, Y. Wang, L. Qian, H. Su, A fixed-time distributed optimization algorithm based on dynamic event-triggered strategy, in *2025 4th Conference on Fully Actuated System Theory and Applications (FASTA)*, IEEE, Piscataway, NJ, 2025, pp. 811–816, `https://doi.org/10.1109/FASTA65681.2025.11138611`.
26. Z. Yang, X. Pan, Q. Zhang, Z. Chen, Distributed optimization for multi-agent systems with time delay, *IEEE Access*, **8**:123019–123025, 2020, `https://doi.org/10.1109/ACCESS.2020.3007731`.

27. Z. Yu, S. Yu, H. Jiang, X. Mei, Distributed fixed-time optimization for multi-agent systems over a directed network, *Nonlinear Dyn.*, **103**(1):775–789, 2021, https://doi.org/10.1007/s11071-020-06116-1.

28. T. Zhao, Z. Ding, Distributed finite-time optimal resource management for microgrids based on multi-agent framework, *IEEE Trans. Ind. Electron.*, **65**(8):6571–6580, 2017, https://doi.org/10.1109/TIE.2017.2721923.

29. Z. Zhao, G. Chen, M. Dai, Distributed event-triggered scheme for a convex optimization problem in multi-agent systems, *Neurocomputing*, **284**:90–98, 2018, https://doi.org/10.1016/j.neucom.2017.12.060.