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Approximate controllability of impulsive neutral fractional hemivariational inequalities involving memory term*

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Abstract. This paper develops a new analytical framework for impulsive neutral fractional differential systems with memory terms. Fractional calculus is adopted for its superior capability in capturing hereditary and memory effects commonly observed in complex dynamical systems. To effectively describe the nonlinear and nonsmooth behavior of the system, the theory of hemivariational inequalities is incorporated. Moreover, history-dependent operators are introduced to explicitly account for the influence of past states on the present dynamics and future evolution of the system. The main objectives of this study are twofold. First, we establish the existence of solutions for the considered impulsive neutral fractional differential systems by employing techniques from fixed-point theory, fractional calculus, and the generalized Clarke subdifferential. Second, we analyze the controllability of the system, with an emphasis on steering the system between prescribed states within a finite time horizon.

Keywords: neutral fractional differential system, memory term, hemivariational inequalities, existence, controllability.

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1 Introduction

Fractional calculus, which extends the concepts of differentiation and integration to non-integer orders, has garnered increasing attention from mathematicians in recent years. In addition, fractional calculus has proven surprisingly effective at modeling real-world phenomena across a range of fields, from engineering and physics to the social sciences and even image processing. Fractional differential equations have emerged as a powerful mathematical framework, frequently offering more accurate representations of complex phenomena than their integer-order counterparts. For further exploration of this topic, readers may refer to key texts [3, 32] as well as relevant articles [7, 8, 24]. As is well known, impulsive differential equations are a core mathematical tool for characterizing the coupling mechanism between continuity and instantaneous jumps in complex evolving systems. In practical scenarios such as sudden disturbances in biological populations, intermittent corrections of satellite orbits, and drug metabolism in the body, systems are often subjected to impulsive perturbations at specific moments, causing their states to undergo instantaneous jumps. Compared with conventional models, the discontinuity of their solutions at impulse points gives the systems more complex dynamical behavior and poses challenges for qualitative analysis. Currently, this field is becoming a major focus in nonlinear dynamics and control theory, and improving its theoretical framework has significant academic value for enhancing the accuracy and robustness of complex system modeling. In recent years, as impulsive differential equations have been studied in greater depth, substantial progress has been made in both theory and applications; see, for example, references [1, 19].

Recent years have witnessed a notable increase in research on hemivariational inequalities (HIs), a methodological approach introduced by Panagiotopoulos (1981) [16, 17]. It provides a robust mathematical framework for handling systems governed by non-smooth and nonconvex energy potentials, which arise commonly in numerous scientific and engineering contexts. HIs bridge two established analytical frameworks: they connect variational inequalities, which address smooth systems, to complementarity problems, commonly used to model friction and contact mechanics. Notably, substantial theoretical advances have been achieved in the mathematical analysis of HIs; see [2,10,12,14,29,31]. Consider a dynamical system whose evolution is governed not only by its present state but also by the lasting effects of its past. Such systems are naturally modeled by neutral differential equations, a significant and mathematically sophisticated class of functional differential equations that extend classical differential equations. Unlike classical models, neutral differential equations embed the system's historical behavior directly into the equation itself, offering deeper and more subtle insights into real-world dynamics. In recent years, significant progress has been made in neutral differential equations, both in theoretical and applied aspects; see [4, 5, 9, 25, 33].

History-dependent operators serve as a cornerstone in the fields of functional analysis and differential equations. Sofonea and Matei [20, 21] introduced these operators to address the analysis of dynamic contact problems characterized by long-term memory effects. Incorporating memory dependence, these operators ensure that the system evolution is governed by both its current configuration and its past trajectory. These operators are

particularly effective for modeling quasistatic contact problems in elastic or viscoelastic materials, covering both frictional and frictionless interactions; see [13,27]. Furthermore, these operators are widely applied across various fields, including the study of mechanical impacts, electrical circuit models involving ideal diodes, and Coulomb friction scenarios.

Control theory is concerned with the influence of external inputs on the evolution of dynamical systems. One of its fundamental problems is to determine whether a system can be driven from an arbitrary initial state to a desired final state by means of admissible controls. The analysis of controllability characterizes the inherent capabilities and limitations of the system and plays a crucial role in the development of viable control strategies. Numerous studies have been developed based on HIs on approximate and exact controllability; see the articles [23,30] and the references therein.

However, the solvability of abstract systems that combine a parabolic impulsive HI with a Caputo fractional evolution inclusion involving history-dependent operators has not yet been addressed. By utilizing fractional calculus, semigroup theory, generalized pseudomonotone operators, and fixed-point techniques in abstract frameworks, we will establish the existence of the solutions to the following systems:

$$\begin{aligned}
&\big\langle {}^{\mathcal{C}}\mathcal{D}_0^{\varsigma}\big[\widetilde{\mu}(\sigma)-\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big],\,\vartheta\big\rangle\\
&\quad\leqslant\big\langle\widetilde{\mathcal{A}}\big[\widetilde{\mu}(\sigma)-\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big],\,\vartheta\big\rangle\\
&\qquad+\big\langle(\mathcal{R}_1\widetilde{\mu})(\sigma)+\varphi\big(\sigma,(\mathcal{R}_2\widetilde{\mu})(\sigma),\widetilde{\mu}(\sigma)\big)+\mathcal{B}\theta(\sigma),\,\vartheta\big\rangle\\
&\qquad+\mathcal{J}^0\big(\sigma,\widetilde{\mu}(\sigma);\vartheta\big)\quad\text{for a.e. }\sigma\in[0,T],\ \sigma\neq\sigma_i,\ \vartheta\in\mathcal{U},\ i=1,2,\dots,m,\\
&\widetilde{\mu}(0)=\widetilde{\mu}_0,\qquad \Delta\widetilde{\mu}(\sigma_i)=\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big),\quad i=1,2,\dots,m.
\end{aligned}\tag{1}$$

Consider the evolution of a state function $\widetilde{\mu}(\cdot)$ in a real, separable Hilbert space $(\mathcal{U},\|\cdot\|_{\mathcal{U}})$. The system dynamics are governed by the Caputo fractional derivative of order $\varsigma\in(0,1)$, denoted by ${}^{\mathcal{C}}\mathcal{D}_0^{\varsigma}$. The operator $\widetilde{\mathcal{A}}:\mathcal{D}(\widetilde{\mathcal{A}})\subseteq\mathcal{U}\to\mathcal{U}$ serves as the infinitesimal generator of a $\mathcal{C}_0$-semigroup $\{S(\sigma)\}_{\sigma\geqslant 0}$ on $\mathcal{U}$. The mapping $\mathcal{J}(\sigma,\cdot):\mathcal{U}\to\mathbb{R}$, which is locally Lipschitz in its second argument, possesses a corresponding Clarke subdifferential. This subdifferential is denoted functionally as $\mathcal{J}^0(\sigma,\cdot;0)$. Let the functions $\mathcal{F}:[0,T]\times\mathcal{U}\to\mathcal{U}$, $\varphi:[0,T]\times\mathcal{U}\times\mathcal{U}\to\mathcal{U}$ be given. $\mathcal{R}_1$ and $\mathcal{R}_2$ are the history-dependent operators. The control function is defined as an element $\theta\in\mathcal{L}^2([0,T],\mathcal{W})$. The space of admissible controls, denoted by $\mathcal{W}$, is endowed with a Hilbert space structure. The operator $\mathcal{B}:\mathcal{W}\to\mathcal{U}$ is bounded and linear, and $\mathcal{L}_i:\mathcal{U}\to\mathcal{U}$ are impulsive functions. Let $0=\sigma_0<\sigma_1<\sigma_2<\cdots<\sigma_m<\sigma_{m+1}=T$ be fixed points, and let the symbol $\Delta\varsigma(\sigma)$ represent the jump in the state ς at σ, which is defined by $\Delta\varsigma(\sigma)=\varsigma(\sigma^+)-\varsigma(\sigma^-)$.

We briefly describe the objectives of this study.

As discussed in the aforementioned literature, many practical systems in physics, engineering, and biology, such as viscoelastic contact processes, neural dynamics, and smart material control, exhibit intertwined features including memory and hereditary effects, nonsmooth and nonconvex nonlinearities, history dependence, instantaneous disturbances, and external control requirements. Fractional derivatives and history-dependent operators provide effective tools for modeling the influence of past states on current

dynamics, while neutral terms enhance modeling flexibility by coupling the system evolution with nonlinear state transformations. Moreover, HIs offer a natural framework for treating nonsmooth and nonconvex energy potentials that cannot be adequately handled by classical differential equations or variational inequalities. Impulse conditions further enable the description of sudden state changes caused by shocks or switching events. These considerations collectively motivate the formulation of system (1) as a unified mathematical model capable of capturing such complex dynamical behaviors.

It is worth noting that the following system has been studied by scholars:

$$\begin{aligned}&\big\langle {}^{\mathcal{C}}\mathcal{D}_0^{\varsigma}\big[\widetilde{\mu}(\sigma)-\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big],\,\vartheta\big\rangle\\&\quad\leqslant\big\langle\widetilde{\mathcal{A}}\big[\widetilde{\mu}(\sigma)-\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big],\,\vartheta\big\rangle+\big\langle\varphi\big(\sigma,(\mathcal{R}\widetilde{\mu})(\sigma),\widetilde{\mu}(\sigma)\big)+\mathcal{B}\theta(\sigma),\,\vartheta\big\rangle\\&\qquad+\mathcal{J}^0(\sigma,\widetilde{\mu}(\sigma);\vartheta)\quad\text{for a.e. }\sigma\in[0,T]\text{ and }\vartheta\in\mathcal{U}.\end{aligned}$$

In addition, we investigate the controllability of the system, which refers to its ability to be driven from one state to another within a specified time period.

This paper proceeds in the following manner. Section 2 presents the foundational framework, including key lemmas, basic notation, related results. Subsequently, Section 3 is devoted to establishing the existence of a mild solution for system (1) by integrating methods from semigroup theory, fractional calculus, and fixed-point theory. As detailed in Section 4, the controllability of our system is thoroughly examined.

2 Preliminaries

This section provides the necessary background in fractional calculus, including key definitions and theorems that will be used in subsequent proofs. For more information, please refer to the literature [7, 24, 26, 28].

Let $\mathcal{Z}$ be a Banach space with norm $\|\cdot\|_{\mathcal{Z}}$, and let $\mathcal{Z}^*$ denote its topological dual. Denote by $\mathcal{C}(I;\mathcal{Z})$ the space of all continuous functions from the interval $I\subset\mathbb{R}$ into $\mathcal{Z}$. This space is endowed with the usual supremum norm $\|\mu\|=\sup_{t\in I}\|\mu(t)\|_{\mathcal{Z}}$, under which $\mathcal{C}(I;\mathcal{Z})$ becomes a Banach space. Define the space $\mathcal{PC}=\mathcal{PC}(I;\mathcal{Z})$ as the set of all functions ϖ such that $\varpi\in\mathcal{C}(I;\mathcal{Z})$, and for a partition of the interval I into subintervals $I_i=(\sigma_i,\sigma_{i+1}]$, where $i=0,1,2,\ldots,m$, the restriction $\varpi|_{I_i}\in\mathcal{C}(I_i;\mathcal{Z})$. Additionally, the left and right limits $\varpi(\sigma_i^-)$ and $\varpi(\sigma_i^+)$ must exist for every $i=1,2,\ldots,m$. Likewise, given a Banach space $\mathcal{Y}$, we denote the following: $\mathcal{N}_f^c(\mathcal{Y}):=\{\Omega\subseteq\mathcal{Y}\colon\,\Omega\neq\emptyset,$ is closed and convex$\}$, $\mathcal{N}_w^k(c)(\mathcal{Y}):=\{\Omega\subseteq\mathcal{Y}\colon\,\Omega\neq\emptyset$ is weakly compact and convex$\}$.

Next, let us review some properties of fractional calculus.

Definition 1. (See [3].) The Riemann–Liouville fractional integral of a function G for order $\varsigma>0$ is expressed as $I_0^{\varsigma}G(\sigma)=(1/\Gamma(\varsigma))\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}G(\iota)\,\mathrm{d}\iota$, $\sigma>0$, where $\Gamma(\varsigma)$ represents the gamma function defined by $\Gamma(\varsigma)=\int_0^{\infty}x^{\varsigma-1}\mathrm{e}^{-x}\,\mathrm{d}x$. This representation is valid whenever the integral exists pointwise for every σ in the interval $[0,\infty)$.

Definition 2. (See [3].) For a real-valued function $\widetilde{\mu} : [0,\infty) \to \mathbb{R}$ whose first $n-1$ derivatives are absolutely continuous, the Riemann–Liouville fractional derivative of order $\varsigma > 0$ (with base point zero) is defined by

$$ {}^{\mathcal{C}}\mathcal{D}_0^{\varsigma}\widetilde{\mu}(\sigma) = \frac{1}{\Gamma(n-\varsigma)}\frac{\mathrm{d}^n}{\mathrm{d}\sigma^n}\int_0^{\sigma}\frac{\widetilde{\mu}(\iota)}{(\sigma-\iota)^{\varsigma+1-n}}\,\mathrm{d}\iota, $$

where $n-1 < \varsigma < n$, $\sigma > 0$, and $n \in N$.

Definition 3. (See [3].) The Caputo derivative of order ς for a function $\mu : [0,\infty) \to \mathbb{R}$ can be expressed as

$$ {}^{\mathcal{C}}\mathcal{D}_0^{\varsigma}\mu(\sigma) = {}^{\mathcal{L}}\mathcal{D}^{\varsigma}\left(\mu(\sigma) - \sum_{k=0}^{n-1}\frac{\sigma^k}{k!}\mu^{(k)}(0)\right), \quad n-1 < \varsigma < n,\ \sigma > 0. $$

We begin by introducing the fundamental concepts and key results from multivalued analysis.

Definition 4. (See [6].) For a function $G : \mathcal{Z} \to \mathbb{R}$ that is locally Lipschitz, the notation $G^0(\theta;\vartheta)$ refers to the generalized directional derivative of G at the point $\theta \in \mathcal{Z}$ in the direction of $\vartheta \in \mathcal{Z}$, and it is defined by

$$ G^0(\theta;\vartheta) = \limsup_{\kappa \to 0^+,\ \mu \to \theta}\frac{G(\mu+\kappa\vartheta) - G(\mu)}{\kappa}. $$

The generalized gradient of $G : \mathcal{Z} \to \mathbb{R}$ at the point $\kappa \in \mathcal{Z}$ is characterized by

$$ \partial G(\kappa) = \left\{\sigma \in \mathcal{Z}^*\colon\ G^0(\kappa;\zeta) \geqslant \langle \sigma, \zeta\rangle \text{ for all } \zeta \in \mathcal{Z}\right\}. $$

Proposition 1. (*See* [6].) *For an open set $\Omega \subset \mathcal{Z}$, a function $G : \Omega \to \mathbb{R}$ satisfies a local Lipschitz condition on Ω. Then the following statements are true*:

(i) *For every $\vartheta \in \mathcal{Z}$, $G^0(\varpi;\vartheta) = \max\{\langle \varpi^*, \vartheta\rangle,\ \varpi^* \in \partial G(\varpi)\}$;*

(ii) *For every $\varpi \in \Omega$, the subdifferential $\partial G(\varpi)$ is nonempty, convex, and weak*-compact in the dual space $\mathcal{Z}^*$. Moreover, each subgradient $\varpi^* \in \partial G(\varpi)$ satisfies $\|\varpi^*\|_{\mathcal{Z}^*} \leqslant \mathcal{K}$, where $\mathcal{K}$ is the Lipschitz constant of G near ϖ;*

(iii) *The set corresponding to the generalized gradient ∂G is closed in the topological product space $\Omega \times \mathcal{Z}^*_{w^*}$, where $\mathcal{Z}^*_{w^*}$ carries the weak* topology. In precise terms, let $\{\varpi_n\} \subset \mathcal{Z}$ and $\{\varpi_n^*\} \subset \mathcal{Z}^*$ be sequences such that $\varpi_n^* \in \partial G(\varpi_n)$. If*

$$ \varpi_n \to \varpi \quad \text{in } \mathcal{Z}, \qquad \varpi_n^* \xrightarrow{\varpi^*} \varpi^* \quad \text{in } \mathcal{Z}^*, $$

then the limit satisfies $\varpi^* \in \partial G(\varpi)$;

(iv) *the multifunction Ω is such that the subdifferential mapping $\partial G(\cdot)$ is upper semicontinuous when $\mathcal{Z}^*$ is equipped with the weak*-topology.*

Next, define the set-valued operator $\mathcal{N}: L^2(I,\mathcal{U}) \to 2^{L^2(I,\mathcal{U})}$ by

$$\mathcal{N}(\widetilde{\mu}) = \big\{k \in L^2(I,\mathcal{U}):\ k(\sigma) \in \partial\mathcal{J}\big(\sigma,\widetilde{\mu}(\sigma)\big) \text{ for a.e. } \sigma \in I\big\}, \quad \widetilde{\mu} \in L^2(I,\mathcal{U}).$$

Following the works of [23], a mild solution to system (1) can be defined as follows.

Definition 5. For a given control $\theta \in L^2(I,\mathcal{W})$, a function $\widetilde{\mu} \in \mathcal{PC}(I,\mathcal{U})$ is called a *mild solution* of system (1) if it satisfies $\widetilde{\mu}(0) = \widetilde{\mu}_0$; there exists a $J \in \mathcal{L}^1(I,\mathcal{U})$ such that $J(\sigma) \in \partial\mathcal{J}(\sigma,\widetilde{\mu}(\sigma))$ for almost every $t \in I$ and

$$\begin{aligned}\widetilde{\mu}(\sigma) = {}& \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\\ &+ \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota\\ &+ \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\\ &+ \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota\\ &+ \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\\ &+ \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big),\end{aligned}$$

where

$$\widetilde{S}(\sigma) = \int_0^\infty \Psi_\varsigma(\tau)S\big(\sigma^\varsigma\tau\big)\,\mathrm{d}\tau, \qquad \delta(\sigma) = \varsigma\int_0^\infty \tau\Psi_\varsigma(\tau)S\big(\sigma^\varsigma\tau\big)\,\mathrm{d}\tau,$$

$$\Psi_\varsigma(\tau) = \frac{1}{\varsigma}\tau^{-1-1/\varsigma}\Upsilon_\varsigma\big(\tau^{-1/\varsigma}\big) \geqslant 0, \qquad \int_0^\infty \Psi_\varsigma(\tau)\,\mathrm{d}\tau = 1,$$

$$\Upsilon_\varsigma = \frac{1}{\pi}\sum_{n=1}^\infty (-1)^{n-1}\tau^{-\varsigma n-1}\frac{\Gamma(n\varsigma+1)}{n!}\sin(n\pi\varsigma), \quad 0 < \tau < +\infty.$$

Let us now introduce the following lemma, which will be used frequently.

Lemma 1. (*See* [23, *Lemma* 2.10].) *The operators $\widetilde{S}$ and δ have the following properties*:

(i) *For every $\mu \in \mathcal{U}$, $\|\delta(\sigma)\mu\| \leqslant (\varsigma\mathcal{M}/\Gamma)(1+\varsigma)\|\mu\|$ and $\|\widetilde{S}(\sigma)\mu\| \leqslant \mathcal{M}\|\mu\|$*;
(ii) *For all $\sigma > 0$, both $\widetilde{S}(\sigma)$ and $\delta(\sigma)$ are compact operators*;
(iii) *The families $\{\widetilde{S}(\sigma),\ \sigma \geqslant 0\}$ and $\{\delta(\sigma),\ \sigma \geqslant 0\}$ are strongly continuous.*

To establish the existence of a mild solution, we introduce the following assumptions.

($\mathrm{H}_{\tilde{\mathcal{A}}}$) Let $\{S(\sigma)\}_{\sigma\geqslant 0}$ be a $\mathcal{C}_0$-semigroup on a Hilbert space $\mathcal{U}$ such that the operator $S(\sigma)$ is compact for every $\sigma \geqslant 0$.

($\mathrm{H}_{\mathcal{F}}$) The function $\mathcal{F} : I \times \mathcal{U} \to \mathcal{U}$ is continuous and such that

$$\big\|\mathcal{F}(\sigma,\varrho)-\mathcal{F}(\sigma,\chi)\big\| \leqslant \mathcal{L}_{\mathcal{F}}\|\varrho-\chi\|, \qquad \big\|\mathcal{F}\big(\sigma,\varrho(\sigma)\big)\big\| \leqslant M_{\mathcal{F}}\big(1+\big\|\varrho(\sigma)\big\|\big)$$

for $\sigma \in I$, $\varrho, \chi \in \mathcal{U}$ with constants $\mathcal{L}_{\mathcal{F}}, \mathcal{M}_{\mathcal{F}} > 0$.

(H_{φ}) $\varphi : I \times \mathcal{U} \times \mathcal{U} \to \mathcal{U}$ is such that

(i) $\|\varphi(\sigma,0,0)\| \leqslant \zeta(\sigma)$ with $\zeta \in \mathcal{L}^{1/p}(I,\mathbb{R}_+), p \in (0,\varsigma)$,

(ii) $\|\varphi(\sigma,\theta_1,\mu_1)-\varphi(\sigma,\theta_2,\mu_2)\| \leqslant \mathcal{L}_{\varphi}(\|\theta_1-\theta_2\|+\|\mu_1-\mu_2\|)$ for $\sigma \in I$ and $\theta_i, \mu_i \in \mathcal{U}$, $i = 1,2$, with constant $\mathcal{L}_{\varphi} > 0$.

($\mathrm{H}_{\mathcal{R}_i}$) $L^2(I,\mathcal{U}) \to L^2(I,\mathcal{U})$ are history-dependent operators, i.e.,

$$\big\|(\mathcal{R}_i\mu_1)(\sigma)-(\mathcal{R}_i\mu_2)(\sigma)\big\|_{\mathcal{U}} \leqslant \mathcal{L}_{\mathcal{R}_i}\int_0^{\sigma}\big\|\mu_1(s)-\mu_2(s)\big\|_{\mathcal{U}}\,\mathrm{d}s$$

for $\sigma \in I$ all $\mu_i \in \mathcal{U}$ with constants $\mathcal{L}_{\mathcal{R}_i} > 0$, $i = 1,2$.

If we denote $(\mathcal{R}_i 0)(\sigma)$ by $F_i(\sigma)$, $i = 1,2$, then it follows that $F_i(\sigma) \in L^{1/\lambda}(0,R^+)$ for $\lambda \in (0,\varsigma)$. Moreover, for almost every $\sigma \in I$ and every $\mu \in \mathcal{U}$, the following estimate holds:

$$\big\|(\mathcal{R}_i\mu)(\sigma)\big\|_{\mathcal{U}} \leqslant F_i(\sigma)+\mathcal{L}_{\mathcal{R}_i}\int_0^{\sigma}\|\mu(s)\|_{\mathcal{U}}\,\mathrm{d}s.$$

For the properties of history-dependent operators, we refer the reader to [11].

($\mathrm{H}_{\mathcal{J}}$) $\mathcal{J} : I \times \mathcal{U} \to \mathbb{R}$ is such that

(i) $\mathcal{J}(\cdot,\zeta)$ is measurable for all $\zeta \in \mathcal{U}$,

(ii) $\mathcal{J}(\sigma,\cdot)$ is locally Lipschitz continuous for $\sigma \in I$,

(iii) $\|\partial\mathcal{J}(\sigma,\mu)\| = \sup\{\|J\|\colon J(\sigma) \in \partial\mathcal{J}(\sigma,\mu)\} \leqslant \Theta(\sigma)+C_J\|\mu\|$ for $\sigma \in I$ and $\mu \in \mathcal{U}$ with $\Theta \in L^{1/\alpha}(I,\mathbb{R}^+)$, $\alpha \in (0,\varsigma)$, and constant $C_J > 0$.

($\mathrm{H}_{\mathcal{L}_i}$) $\|\mathcal{L}_i(\mu)\| \leqslant d_i$, $i = 1,2,\ldots,m$, for each $\mu \in \mathcal{U}$ with constant $d_i > 0$.

In the final section, we provide an important fixed-point theorem; see [23, Thm. 2.12] and [22, Thm. 2.7].

Theorem 1. *Consider a locally convex Banach space $\mathcal{Z}$ and an upper semicontinuous multifunction $\chi : \mathcal{Z} \to 2^{\mathcal{Z}}$ with compact convex values. Assume that there is a closed neighborhood $\mathcal{O}$ of 0 for which the image $\chi(\mathcal{O})$ is relatively compact. Then, provided that the set $\Omega = \{\zeta \in \mathcal{Z}\colon \exists\lambda > 1$ such that $\lambda\zeta \in \chi(\zeta)\}$ is bounded, χ possesses a fixed point.*

3 Existence results

The core analytical task of this section is to derive existence results for the mild solution framework of system (1).

Theorem 2. *Provided that*

$$\mathcal{M}M_{\mathcal{F}} + \frac{\mathcal{L}_{\mathcal{R}_1}T^{1+\varsigma}\mathcal{M}}{\Gamma(1+\varsigma)} + \frac{(\mathcal{L}_{\mathcal{R}_2}T^{1+\varsigma}+T^{\varsigma})\mathcal{M}\mathcal{L}_{\varphi}}{\Gamma(1+\varsigma)} + \frac{C_J\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)} < 1$$

and assuming that hypotheses $(\mathrm{H}_{\mathcal{F}})$, (H_{φ}), $(\mathrm{H}_{\mathcal{R}_i})$, $(\mathrm{H}_{\mathcal{J}})$, *and* $(\mathrm{H}_{\mathcal{L}_i})$ *are satisfied, system* (1) *admits at least one mild solution over the domain* I *for any admissible control input* θ *taken from the space* $\mathcal{L}^2(I,\mathcal{W})$.

Proof. Initially, for any $\widetilde{\mu} \in \mathcal{PC}(I,\mathcal{U}) \subset L^2(I,\mathcal{U})$, by Definition 5, we may define the multivalued map $\chi : \mathcal{PC}(I,\mathcal{U}) \to 2^{\mathcal{PC}(I,\mathcal{U})}$ by

$$\chi(\widetilde{\mu}) = \big\{G_J(\widetilde{\mu}),\ J \in \mathcal{N}(\widetilde{\mu})\big\},$$

where

$$\begin{aligned}
G_J(\widetilde{\mu})(\sigma) = {}& \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big) \\
&+ \int_0^{\sigma} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\
&+ \int_0^{\sigma} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\
&+ \int_0^{\sigma} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \\
&+ \int_0^{\sigma} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \\
&+ \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big), \quad J \in \mathcal{N}(\widetilde{\mu}).
\end{aligned}$$

It follows from the convexity of $\mathcal{N}(\widetilde{\mu})$ that $\chi(\widetilde{\mu})$ is also convex. Consequently, the existence of a fixed point for χ is guaranteed by Theorem 1.

We now proceed to establish the proof through the following sequence of steps.

Step 1. For each $\varrho > 0$, the set $\chi(\mathcal{B}_\varrho)$ remains bounded in the space $\mathcal{PC}(I,\mathcal{U})$. Here $\mathcal{B}_\varrho = \{\widetilde{\mu} \in \mathcal{PC}(I,\mathcal{U})\colon \|\widetilde{\mu}\|_{\mathcal{PC}} \leqslant \varrho\}$. In fact, it suffices to show that there exists a constant $L > 0$ such that, for each $\kappa \in \chi(\widetilde{\mu})$, $\widetilde{\mu} \in \mathcal{B}_\varrho$, we have $\|\kappa\|_{\mathcal{PC}} \leqslant L$. If $\kappa \in \chi(\widetilde{\mu})$, then we

can find $J \in \mathcal{N}(\widetilde{\mu})$ such that

$$\begin{aligned}\kappa(\sigma) &= \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0, \widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma, \widetilde{\mu}(\sigma)\big) \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \\ &\quad + \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big), \quad \sigma \in I.\end{aligned}$$

By the Hölder inequality, we deduce

$$\begin{aligned}\big\|\kappa(\sigma)\big\| &\leqslant \mathcal{M}\|\widetilde{\mu}_0\| + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big) + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}(\sigma)\|\big) \\ &\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\left(F_1(\iota) + \mathcal{L}_{\mathcal{R}_1}\int_0^\iota \|\mu(s)\|_{\mathcal{U}}\,\mathrm{d}s\right)\mathrm{d}\iota \\ &\quad + \frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\left(\big\|\varphi(\iota,0,0)\big\| + \left(F_2(\iota) + \mathcal{L}_{\mathcal{R}_2}\int_0^\iota \big\|\mu(s)\big\|_{\mathcal{U}}\,\mathrm{d}s\right)\right. \\ &\qquad \left. + \big\|\widetilde{\mu}(\iota)\big\|\right)\mathrm{d}\iota \\ &\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\|\mathcal{B}\|\big\|\theta(\iota)\big\|\,\mathrm{d}\iota \\ &\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\big(\Theta(\iota) + C_J\big\|\mu(\iota)\big\|\big)\,\mathrm{d}\iota + \mathcal{M}\sum_{i=1}^m d_i \\ &\leqslant \mathcal{M}\|\widetilde{\mu}_0\| + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big) + \mathcal{M}M_{\mathcal{F}}(1+\varrho) \\ &\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\big(F_1(\iota) + \mathcal{L}_{\mathcal{R}_1}\varrho T\big)\,\mathrm{d}\iota \\ &\quad + \frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\big(\big\|\varphi(\iota,0,0)\big\| + F_2(\iota) + \mathcal{L}_{\mathcal{R}_2}\varrho T + \big\|\widetilde{\mu}(\iota)\big\|\big)\,\mathrm{d}\iota\end{aligned}$$

$$
\begin{aligned}
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\|\mathcal{B}\|\big\|\theta(\iota)\big\|\,\mathrm{d}\iota\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\big(\Theta(\iota)+C_J\varrho\big)\,\mathrm{d}\iota+\mathcal{M}\sum_{i=1}^m d_i\\
&\leqslant \mathcal{M}\|\widetilde{\mu}_0\|+\mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big)+\mathcal{M}M_{\mathcal{F}}(1+\varrho)\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\varrho T^{1+\varsigma}}{\varsigma}\right)\\
&+\frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\right.\\
&\left.+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_2}\varrho T^{1+\varsigma}+\varrho T^\varsigma}{\varsigma}\right)\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\frac{1}{(2\varsigma-1)^{1/2}}T^{(2\varsigma-1)/2}\|\mathcal{B}\|\|\theta\|_{\mathcal{L}^2(I,\mathcal{W})}\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\varrho\mathcal{M}T^\varsigma}{\Gamma(1+\varsigma)}+\mathcal{M}\sum_{i=1}^m d_i.
\end{aligned}
$$

Therefore, the $\chi(\mathcal{B}_\varrho)$ remains bounded in the space $\mathcal{PC}(I,\mathcal{U})$.

Step 2. $\{\chi(\widetilde{\mu}):\widetilde{\mu}\in\mathcal{B}_\varrho\}$ is equicontinuous.

If $\kappa\in\chi(\widetilde{\mu})$, then we can find $J\in\mathcal{N}(\widetilde{\mu})$ such that

$$
\begin{aligned}
\big\|\kappa(\sigma)\big\|\leqslant{}&\big\|\widetilde{S}(\sigma)\big(\widetilde{\mu}_0-\mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big)\big\|+\big\|\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big\|\\
&+\left\|\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota\right\|\\
&+\left\|\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\right\|\\
&+\left\|\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota\right\|\\
&+\left\|\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\right\|\\
&+\left\|\sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big)\right\|.
\end{aligned}
$$

It follows from $\sigma_1, \sigma_2 > 0$ that

$$
\begin{aligned}
&\big\|\kappa(\sigma_2) - \kappa(\sigma_1)\big\| \\
&\quad\leqslant \big\|\widetilde{S}(\sigma_2) - \widetilde{S}(\sigma_1)\big\|\big\|\widetilde{\mu}_0 - \mathcal{F}\big(0, \widetilde{\mu}(0)\big)\big\| + \big\|\mathcal{F}\big(\sigma_2, \widetilde{\mu}(\sigma_2)\big) - \mathcal{F}\big(\sigma_1, \widetilde{\mu}(\sigma_1)\big)\big\| \\
&\qquad+ \left\| \int_0^{\sigma_1} \big[(\sigma_2 - \iota)^{\varsigma-1} - (\sigma_1 - \iota)^{\varsigma-1}\big]\delta(\sigma_2 - \iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} (\sigma_1 - \iota)^{\varsigma-1}\big[\delta(\sigma_2 - \iota) - \delta(\sigma_1 - \iota)\big](\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - \iota)^{\varsigma-1}\delta(\sigma_2 - \iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} \big[(\sigma_2 - \iota)^{\varsigma-1} - (\sigma_1 - \iota)^{\varsigma-1}\big]\delta(\sigma_2 - \iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} (\sigma_1 - \iota)^{\varsigma-1}\big[\delta(\sigma_2 - \iota) - \delta(\sigma_1 - \iota)\big]\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - \iota)^{\varsigma-1}\delta(\sigma_2 - \iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} \big[(\sigma_2 - \iota)^{\varsigma-1} - (\sigma_1 - \iota)^{\varsigma-1}\big]\delta(\sigma_2 - \iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} (\sigma_1 - \iota)^{\varsigma-1}\big[\delta(\sigma_2 - \iota) - \delta(\sigma_1 - \iota)\big]\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - \iota)^{\varsigma-1}\delta(\sigma_2 - \iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} \big[(\sigma_2 - \iota)^{\varsigma-1} - (\sigma_1 - \iota)^{\varsigma-1}\big]\delta(\sigma_2 - \iota)J(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_0^{\sigma_1} (\sigma_1 - \iota)^{\varsigma-1}\big[\delta(\sigma_2 - \iota) - \delta(\sigma_1 - \iota)\big]J(\iota)\,\mathrm{d}\iota \right\| \\
&\qquad+ \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2 - \iota)^{\varsigma-1}\delta(\sigma_2 - \iota)J(\iota)\,\mathrm{d}\iota \right\|
\end{aligned}
$$

$$+\left\|\sum_{0<t_i<\sigma_1}\widetilde{S}(\sigma_2-\sigma_i)-\widetilde{S}(\sigma_1-\sigma_i)\mathcal{L}_i(\widetilde{\mu}(t_i))\right\|+\left\|\sum_{\sigma_1<t_i<\sigma_2}\widetilde{S}(\sigma_2-t_i)\mathcal{L}_i\big(\widetilde{\mu}(t_i)\big)\right\|$$
$$=\sum_{n=1}^{7}\Pi_n. \tag{2}$$

From condition ($\mathrm{H}_{\mathcal{F}}$) we deduce

$$\begin{aligned}\Pi_2&\leqslant\big\|\mathcal{F}\big(\sigma_2,\widetilde{\mu}(\sigma_2)\big)-\mathcal{F}\big(\sigma_1,\widetilde{\mu}(\sigma_2)\big)\big\|+\big\|\mathcal{F}\big(\sigma_1,\widetilde{\mu}(\sigma_2)\big)-\mathcal{F}\big(\sigma_1,\widetilde{\mu}(\sigma_1)\big)\big\|\\&\leqslant\big\|\mathcal{F}\big(\sigma_2,\widetilde{\mu}(\sigma_2)\big)-\mathcal{F}\big(\sigma_1,\widetilde{\mu}(\sigma_2)\big)\big\|+\mathcal{L}_{\mathcal{F}}\big\|\widetilde{\mu}(\sigma_2)-\widetilde{\mu}(\sigma_1)\big\|.\end{aligned}$$

By the Hölder inequality, we infer that

$$\begin{aligned}\Pi_3&\leqslant\left\|\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]\delta(\sigma_2-\iota)\big(F_1(\iota)+\mathcal{L}_{\mathcal{R}_1}\int_0^{\iota}\|\mu(s)\|_{\mathcal{U}}\,\mathrm{d}s\big)\,\mathrm{d}\iota\right\|\\&\quad+\left\|\int_0^{\sigma_1}(\sigma_1-\iota)^{\varsigma-1}\big[\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big]\left(F_1(\iota)+\mathcal{L}_{\mathcal{R}_1}\int_0^{\iota}\big\|\mu(s)\big\|_{\mathcal{U}}\,\mathrm{d}s\right)\mathrm{d}\iota\right\|\\&\quad+\left\|\int_{\sigma_1}^{\sigma_2}(\sigma_2-\iota)^{\varsigma-1}\delta(\sigma_2-\iota)\big(F_1(\iota)+\mathcal{L}_{\mathcal{R}_1}\int_0^{\iota}\|\mu(s)\|_{\mathcal{U}}\,\mathrm{d}s\big)\,\mathrm{d}\iota\right\|\\&\leqslant\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]^{1/(1-\lambda)}\mathrm{d}\iota\right)^{1-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}\\&\quad+\frac{\varrho T\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^{\sigma_1}[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}]\,\mathrm{d}\iota+\sup_{\iota\in I}\big\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big\|\\&\quad\times\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\varrho T^{1+\varsigma}}{\varsigma}\right)\\&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}(\sigma_2-\sigma_1)^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}\right.\\&\quad\left.+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}(\sigma_2-\sigma_1)^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\varrho T(\sigma_2-\sigma_1)^{\varsigma}}{\varsigma}\right).\end{aligned}\tag{3}$$

Similar to the proof of inequality (3), we obtain

$$\begin{aligned}\Pi_4\leqslant\Bigg\|\int_0^{\sigma_1}&\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]\delta(\sigma_2-\iota)\\&\times\big(\big\|\varphi(\iota,0,0)\big\|+\mathcal{L}_{\varphi}\big(F_2(\iota)+\mathcal{L}_{\mathcal{R}_2}\varrho T\big)+\mathcal{L}_{\varphi}\big\|\widetilde{\mu}(\iota)\big\|\big)\,\mathrm{d}\iota\Bigg\|\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^{\sigma_1}(\sigma_1-\iota)^{\varsigma-1}\big[\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big]\right.\\
&\qquad\left.\times\big(\big\|\varphi(\iota,0,0)\big\|+\mathcal{L}_\varphi\big(F_2(\iota)+\mathcal{L}_{\mathcal{R}_2}\varrho T\big)+\mathcal{L}_\varphi\big\|\widetilde{\mu}(\iota)\big\|\big)\,\mathrm{d}\iota\right\|\\
&+\left\|\int_{\sigma_1}^{\sigma_2}(\sigma_2-\iota)^{\varsigma-1}\delta(\sigma_2-\iota)\big(\big\|\varphi(\iota,0,0)\big\|+\mathcal{L}_\varphi\big(F_2(\iota)+\mathcal{L}_{\mathcal{R}_2}\varrho T\big)+\mathcal{L}_\varphi\big\|\widetilde{\mu}(\iota)\big\|\big)\,\mathrm{d}\iota\right\|\\
&\leqslant\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]^{1/(1-p)}\,\mathrm{d}\iota\right)^{1-p}\|\zeta\|_{L^{1/p}(0,R^+)}\\
&+\frac{\mathcal{L}_\varphi\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]^{1/(1-\lambda)}\,\mathrm{d}\iota\right)^{1-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}\\
&+\frac{\mathcal{L}_\varphi\varrho(1+\mathcal{L}_{\mathcal{R}_2}T)\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]\,\mathrm{d}\iota\\
&+\sup_{\iota\in I}\big\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big\|\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\right)\\
&+\sup_{\iota\in I}\big\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big\|\mathcal{L}_\varphi\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_2}\varrho T^{1+\varsigma}}{\varsigma}\right)\\
&+\sup_{\iota\in I}\big\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big\|\frac{\mathcal{L}_\varphi\varrho T^\varsigma}{\varsigma}\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-p}{\varsigma-p}\right]^{1-p}(\sigma_2-\sigma_1)^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\\
&+\frac{\mathcal{L}_\varphi\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}(\sigma_2-\sigma_1)^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}\\
&+\frac{\mathcal{L}_\varphi\varrho(1+\mathcal{L}_{\mathcal{R}_2}T)\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}(\sigma_2-\sigma_1)^{\varsigma-\lambda}.
\end{aligned}
$$

Following the proof of the above inequality, it is not difficult to see

$$
\begin{aligned}
\Pi_5&\leqslant\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\int_0^{\sigma_1}\big[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\big]^2\,\mathrm{d}\iota\right)^{1/2}\|\mathcal{B}\|\|\theta\|_{L^2(0,R^+)}\\
&\quad+\sup_{\iota\in I}\big\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\big\|\left[\frac{1}{2\varsigma-1}\right]^{1/2}T^{\varsigma-1/2}\|\mathcal{B}\|\|\theta\|_{L^2(0,R^+)}\\
&\quad+\left[\frac{1}{2\varsigma-1}\right]^{1/2}(\sigma_2-\sigma_1)^{\varsigma-1/2}\|\mathcal{B}\|\|\theta\|_{L^2(0,R^+)}.
\end{aligned}
$$

According to condition ($\mathrm{H}_{\mathcal{J}}$), we deduce

$$
\begin{aligned}
\Pi_6 &\leqslant \left\| \int_0^{\sigma_1} \left[(\sigma_2-\iota)^{\varsigma-1} - (\sigma_1-\iota)^{\varsigma-1}\right]\delta(\sigma_2-\iota)\big(\Theta(\iota)+C_J\|\widetilde{\mu}(\iota)\|\big)\,\mathrm{d}\iota \right\| \\
&\quad + \left\| \int_0^{\sigma_1} (\sigma_1-\iota)^{\varsigma-1}\left[\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\right]\big(\Theta(\iota)+C_J\|\widetilde{\mu}(\iota)\|\big)\,\mathrm{d}\iota \right\| \\
&\quad + \left\| \int_{\sigma_1}^{\sigma_2} (\sigma_2-\iota)^{\varsigma-1}\delta(\sigma_2-\iota)\big(\Theta(\iota)+C_J\|\widetilde{\mu}(\iota)\|\big)\,\mathrm{d}\iota \right\| \\
&\leqslant \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\int_0^{\sigma_1}\left[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\right]^{1/(1-\alpha)}\mathrm{d}\iota\right)^{1-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)} \\
&\quad + \frac{\varrho C_J\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\int_0^{\sigma_1}\left[(\sigma_2-\iota)^{\varsigma-1}-(\sigma_1-\iota)^{\varsigma-1}\right]\mathrm{d}\iota \\
&\quad + \sup_{\iota\in I}\|\delta(\sigma_2-\iota)-\delta(\sigma_1-\iota)\|\left(\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\varrho T^{\varsigma}}{\varsigma}\right) \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}(\sigma_2-\sigma_1)^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\varrho(\sigma_2-\sigma_1)^{\varsigma}}{\varsigma}\right).
\end{aligned}
$$

By condition ($\mathrm{H}_{\mathcal{L}_i}$), we derive

$$
\Pi_7 \leqslant \left\| \sum_{0<t_i<\sigma_1} (\widetilde{S}(\sigma_2-t_i)-\widetilde{S}(\sigma_1-t_i))d_i \right\| + \mathcal{M}\| \sum_{\sigma_1<t_i<\sigma_2} d_i\|.
$$

By Lemma 1, the family of operators $\{S(\sigma)\}_{\sigma>0}$ is compact. Then it follows from [18] that it is also continuous in the uniform operator topology. Consequently, the term on the right-hand side of (2) tends to zero as $\sigma_2 \to \sigma_1$.

In particular, fixing $\sigma_1 = 0$ and choosing any σ_2 such that $0 < \sigma_2 \leqslant T$, it follows that the quantity $\|\widetilde{\mu}(\sigma_2) - \widetilde{\mu}_0\|_{\mathcal{U}}$ converges to zero as $\sigma_2 \to 0$. This convergence holds uniformly for all $\widetilde{\mu}$ within the ball $\mathcal{B}_\varrho$.

Consequently, it follows that the collection of functions defined by $\{\chi(\widetilde{\mu}) : \widetilde{\mu} \in \mathcal{B}_\varrho\}$ forms an equicontinuous family in the space $\mathcal{PC}(I,\mathcal{U})$.

Step 3. χ is completely continuous.

Define the set

$$
\Delta(\sigma) := \big\{\kappa(\sigma),\ \kappa \in \chi(\widetilde{\mu})\big\}, \quad \sigma \geqslant 0.
$$

We will show that $\Delta(\sigma)$ is relatively compact in $\mathcal{U}$. For $\sigma = 0$, we have $\Delta(0) = \{\widetilde{\mu}_0\}$, which is compact. Therefore, it remains to consider the case $\sigma > 0$. For every $\widetilde{\mu} \in \mathcal{B}_\varrho$

and every $\kappa \in \chi(\widetilde{\mu})$, we can find $J \in \mathcal{N}(\widetilde{\mu})$ such that

$$\begin{aligned}\kappa(\sigma) = {} & \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0, \widetilde{\mu}(0)\big)\big) \\ & + \mathcal{F}\big(\sigma, \widetilde{\mu}(\sigma)\big) + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\ & + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\ & + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \\ & + \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big).\end{aligned}$$

For $\sigma \in (0, T]$ and $\eta \in (0, \sigma)$, we have

$$\begin{aligned}\kappa_\eta(\sigma) = {} & \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0, \widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma, \widetilde{\mu}(\sigma)\big) \\ & + \int_0^{\sigma-\eta} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\ & + \int_0^{\sigma-\eta} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\ & + \int_0^{\sigma-\eta} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota))\,\mathrm{d}\iota + \int_0^{\sigma-\eta} (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \\ & + \sum_{0<\sigma_i<\sigma-\eta} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big), \quad \sigma \in I.\end{aligned}$$

Given the compactness of the operator $S(\sigma)$, it follows that for any $\eta \in (0, \sigma)$, the set

$$\Delta_\eta(\sigma) := \big\{\kappa_\eta(\sigma),\ \kappa \in \chi(\widetilde{\mu})\big\}$$

is relatively compact in $\mathcal{U}$.

Based on the derivation in Step 2, it is not difficult to prove that

$$\begin{aligned}&\big\|\kappa(\sigma) - \kappa_\eta(\sigma)\big\| \\ &\leqslant \int_{\sigma-\eta}^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota + \int_{\sigma-\eta}^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota, (\mathcal{R}_2\widetilde{\mu})(\iota), \widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\end{aligned}$$

$$
\begin{aligned}
&+\int_{\sigma-\eta}^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(t-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota+\int_{\sigma-\eta}^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\\
&+\sum_{\sigma-\eta<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big)\\
&\leqslant\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}\eta^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}\right.\\
&\left.+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}\eta^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\varrho T\eta^{\varsigma}}{\varsigma}\right)\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-p}{\varsigma-p}\right]^{1-p}\eta^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\\
&+\frac{\mathcal{L}_{\varphi}\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}\eta^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}\\
&+\frac{\mathcal{L}_{\varphi}\varrho(1+\mathcal{L}_{\mathcal{R}_2}T)\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}\eta^{\varsigma-\lambda}+\left[\frac{1}{2\varsigma-1}\right]^{1/2}\eta^{\varsigma-1/2}\|\mathcal{B}\|\|\theta\|_{L^2(0,R^+)}\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}\eta^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\varrho\eta^{\varsigma}}{\varsigma}\right)+\mathcal{M}\left\|\sum_{t-\eta<t_i<t}d_i\right\|.
\end{aligned}
$$

This implies that the family $\Delta(\sigma)$ for $\sigma>0$ is totally bounded, and therefore relatively compact in $\mathcal{U}$. Applying the Ascoli–Arzelà theorem, we conclude that the mapping χ is a compact operator.

Step 4. χ has a closed graph.

Assume that $\widetilde{\mu}_n\to\widetilde{\mu}$ in $\mathcal{PC}(I;\mathcal{U})$, $\kappa_n\in\chi(\widetilde{\mu}_n)$ and $\kappa_n\to\kappa$ in $\mathcal{PC}(I;\mathcal{U})$. Then we claim that $\kappa\in\chi(\widetilde{\mu})$. According to $\kappa_n\in\chi(\widetilde{\mu}_n)$, we can find $J_n\in\mathcal{N}(\widetilde{\mu}_n)$ such that

$$
\begin{aligned}
\kappa_n(\sigma)&=\widetilde{S}(\sigma)\big(\widetilde{\mu}_0-\mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big)+\mathcal{F}\big(\sigma,\widetilde{\mu}_n(\sigma)\big)\\
&\quad+\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu}_n)(\iota)\,\mathrm{d}\iota\\
&\quad+\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu}_n)(\iota),\widetilde{\mu}_n(\iota)\big)\,\mathrm{d}\iota\\
&\quad+\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota+\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J_n(\iota)\,\mathrm{d}\iota\\
&\quad+\sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}_n(\sigma_i)\big),\quad\sigma\in I.
\end{aligned}
$$

Based on hypothesis ($\mathrm{H}_{\mathcal{J}}$), it can be established that the sequence $\{J_n\}_{n\geqslant1}\subset L^2(I,\mathcal{U})$ is bounded. Therefore, possibly by extracting a subsequence, we may assume without

loss of generality that J_n converges weakly to some J in $L^2(I,\mathcal{U})$. It follows from the compactness of $S(\sigma)$ that

$$\begin{aligned}\kappa(\sigma) &= \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(t-\iota)\big(\mathcal{R}_1\widetilde{\mu}\big)(\iota)\,\mathrm{d}\iota\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\\ &\quad + \sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big),\quad \sigma\in I.\end{aligned}$$

We can see that $\kappa_n\to\kappa$ in $\mathcal{PC}(I;\mathcal{U})$ and $\kappa_n\in\chi(\widetilde{\mu}_n)$. From Proposition 1 we obtain $J\in\mathcal{N}(\widetilde{\mu})$. Hence, we have proved that $\kappa\in\chi(\widetilde{\mu})$, which implies that Υ_β has a closed graph. By Proposition 3.31(2) of [15], χ is upper semicontinuous.

Step 5. A priori estimate.

Based on the conclusions derived in Steps 1–4, the set-valued map χ is known to have compact convex values and is upper semicontinuous; furthermore, $\chi(\mathcal{B}_\varrho)$ is relatively compact.

In order to apply Theorem 1 and establish the existence of a fixed point for χ, it now suffices to prove that the set

$$\Omega = \big\{\zeta\in\mathcal{U}\colon \exists\lambda>1 \text{ such that } \lambda\zeta\in\chi(\zeta)\big\}$$

is bounded. If $\widetilde{\mu}\in\Omega$, then we can find $J\in\mathcal{N}(\widetilde{\mu})$ such that

$$\begin{aligned}\widetilde{\mu}(\sigma) &= \lambda^{-1}\widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \lambda^{-1}\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\\ &\quad + \lambda^{-1}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota\\ &\quad + \lambda^{-1}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\\ &\quad + \lambda^{-1}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota + \lambda^{-1}\int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\\ &\quad + \lambda^{-1}\sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big),\quad \sigma\in I.\end{aligned}$$

Based on the above conditions, it can be concluded that

$$\begin{aligned}
\big\|\widetilde{\mu}(\sigma)\big\| &\leqslant \big\|\widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big)\big\| + \big\|\mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big)\big\| \\
&\quad + \left\| \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \right\| \\
&\quad + \left\| \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \right\| \\
&\quad + \left\| \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta(\iota)\,\mathrm{d}\iota \right\| + \left\| \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \right\| \\
&\quad + \left\| \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i(\widetilde{\mu}(\sigma_i)) \right\| \\
&\leqslant \mathcal{M}\|\widetilde{\mu}_0\| + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big) + \mathcal{M}M_{\mathcal{F}}\big(1+\big\|\widetilde{\mu}(\sigma)\big\|\big) \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)} + \frac{\mathcal{L}_{\mathcal{R}_1}\|\widetilde{\mu}(\sigma)\|T^{1+\varsigma}}{\varsigma}\right) \\
&\quad + \frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\right. \\
&\quad \left. + \left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)} + \frac{\mathcal{L}_{\mathcal{R}_2}\|\widetilde{\mu}(\sigma)\|T^{1+\varsigma} + \|\widetilde{\mu}(\sigma)\|T^{\varsigma}}{\varsigma}\right) \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\frac{1}{(2\varsigma-1)^{1/2}}T^{(2\varsigma-1)/2}\|\mathcal{B}\|\|\theta\|_{\mathcal{L}^2(I,\mathcal{W})} \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)} + \frac{C_J\|\widetilde{\mu}(\sigma)\|\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)} + \mathcal{M}\sum_{i=1}^{m} d_i \\
&:= \mathcal{C}_1 + \mathcal{C}_2\big\|\widetilde{\mu}(\sigma)\big\|,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}_1 &= \mathcal{M}\|\widetilde{\mu}_0\| + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big) + \mathcal{M}M_{\mathcal{F}} \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)} \\
&\quad + \frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)} + \left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}\right) \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\frac{1}{(2\varsigma-1)^{1/2}}T^{(2\varsigma-1)/2}\|\mathcal{B}\|\|\theta\|_{\mathcal{L}^2(I,\mathcal{W})} \\
&\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)} + \mathcal{M}\sum_{i=1}^{m} d_i
\end{aligned}$$

and

$$\mathcal{C}_2 = \mathcal{M}M_{\mathcal{F}} + \frac{\mathcal{L}_{\mathcal{R}_1}T^{1+\varsigma}\mathcal{M}}{\Gamma(1+\varsigma)} + \frac{(\mathcal{L}_{\mathcal{R}_2}T^{1+\varsigma}+T^{\varsigma})\mathcal{M}\mathcal{L}_{\varphi}}{\Gamma(1+\varsigma)} + \frac{C_J\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)}.$$

Given that $\mathcal{C}_2 < 1$, we have

$$\|\widetilde{\mu}\|_{\mathcal{PC}} \leqslant \frac{\mathcal{C}_1}{1-\mathcal{C}_2}.$$

Thus, the set Ω is bounded. By Theorem 1, operator χ possesses a fixed point, which corresponds to a mild solution of system (1) on the interval I. □

4 Approximate controllability results

Let us now turn to the investigation of approximate controllability for system (1).

Given an initial state $\widetilde{\mu}_0 \in \mathcal{U}$ and an admissible control θ, the solution to system (1) evaluated at the terminal instant T is written as $\widetilde{\mu}_T(\widetilde{\mu}_0;\theta)$.

Define the reachable set at terminal time T from initial state $\widetilde{\mu}_0$ for system (1) as

$$\mathcal{R}(T,\widetilde{\mu}_0) := \big\{\widetilde{\mu}_T(\widetilde{\mu}_0;\theta),\ \theta(\cdot) \in L^2(I,\mathcal{W})\big\}.$$

Let $\overline{\mathcal{R}}(T,\widetilde{\mu}_0)$ represent the closure of this set in the space $\mathcal{U}$. A core definition for our subsequent development is now presented.

Definition 6. System (1) is defined to be approximately controllable on the interval I if, for any initial state $\widetilde{\mu}_0 \in \mathcal{U}$, the corresponding reachable set $\mathcal{R}(T,\widetilde{\mu}_0)$ is dense in the space $\mathcal{U}$.

The linear differential system is governed by

$$^{\mathcal{C}}\mathcal{D}_0^{\varsigma}\widetilde{\mu}(\sigma) \in \widetilde{\mathcal{A}}\widetilde{\mu}(\sigma) + \mathcal{B}\theta(\sigma), \quad \sigma \in I, \tag{4}$$

subject to $\widetilde{\mu}(0) = \widetilde{\mu}_0$.

We now define the controllability and resolvent operators for system (1) as follows:

$$\Upsilon_0^T := \int_0^T \widetilde{S}(T-s)\mathcal{B}\mathcal{B}^*\widetilde{S}^*(T-s)\,\mathrm{d}s.$$

Subsequently, for a given parameter $\beta > 0$, the associated resolvent is given by

$$P\big(\beta,\Upsilon_0^T\big) = \big(\beta I + \Upsilon_0^T\big)^{-1},$$

where $\mathcal{B}^*$ and $\widetilde{S}^*(t)$ represent the adjoint operators of $\mathcal{B}$ and $\widetilde{S}(t)$.

Lemma 2. *A necessary and sufficient condition for the approximate controllability of system* (4) *on I is that $\beta P(\beta,\Upsilon_0^T) \to 0$ in the strong operator topology as $\beta \to 0^+$.*

For any $\beta > 0$, any $\widetilde{\mu} \in \mathcal{PC}(I,\mathcal{U}) \subset L^2(I,\mathcal{U})$, and any $\widetilde{\mu}_1 \in \mathcal{U}$, one can define a set-valued map $\Upsilon_\beta : \mathcal{PC}(I,\mathcal{U}) \to 2^{\mathcal{PC}(I,\mathcal{U})}$ by

$$\Upsilon_\beta = \big\{G_J(\widetilde{\mu}),\ J \in \mathcal{N}(\widetilde{\mu})\big\}$$

with

$$\begin{aligned} G_J(\sigma) &= \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma,\widetilde{\mu}(\sigma)\big) \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta_\beta(\iota)\,\mathrm{d}\iota + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota \\ &\quad + \sum_{0<\sigma_i<\sigma} \widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big), \quad J \in \mathcal{N}(\widetilde{\mu}), \end{aligned}$$

where

$$\begin{aligned} \theta_\beta(\sigma) &= \mathcal{B}^*\widetilde{S}^*(T-s)P\big(\beta,\Upsilon_0^T\big)\bigg[\widetilde{\mu}_1 - \widetilde{S}(T)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) \\ &\quad - \mathcal{F}\big(T,\widetilde{\mu}(T)\big) - \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\ &\quad - \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\ &\quad - \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)\mathcal{B}\theta_\beta(\iota)\,\mathrm{d}\iota - \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)J(\iota)\,\mathrm{d}\iota \\ &\quad - \sum_{0<\sigma_i<T} \widetilde{S}(T-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big)\bigg]. \end{aligned}$$

Theorem 3. *For every $\beta > 0$, the mapping Υ_β possesses a fixed point in the interval I, provided that the following condition holds*:

$$\begin{aligned} &\bigg(1 + \frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\bigg)\bigg(\mathcal{M}\|\widetilde{\mu}_0\| + \mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big) + \mathcal{M}M_{\mathcal{F}} \\ &\quad + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\bigg(\bigg[\frac{1-\lambda}{\varsigma-\lambda}\bigg]^{1-\lambda} T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}\bigg) \end{aligned}$$

$$+\frac{\varsigma\mathcal{M}\mathcal{L}_{\varphi}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^{+})}+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^{+})}\right)$$
$$+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-p}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^{+})}+\mathcal{M}\sum_{i=1}^{m}d_i\Bigg)+\frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\|\widetilde{\mu}_1\|$$
$$<1.$$

Proof. Following Theorem 2, the multimap Υ_β is well defined under the control θ_β and the specified assumptions. To obtain the results, it therefore suffices to verify that Υ_β satisfies all hypotheses of Theorem 2. The argument proceeds similarly to the proof of Theorem 2 and is omitted for conciseness. Only the fifth step of the verification is included below.

Step 5. A priori estimate.

From the fact that Υ_β is compact and convex-valued and building on the results established in Steps 1–4, one concludes that $\Upsilon_\beta(\mathcal{B}_\varrho)$ forms a relatively compact set with respect to upper semicontinuity. In order to apply Theorem 1 and establish the existence of a fixed point for χ, it now suffices to prove that the set

$$\Omega_\beta=\big\{\zeta\in\mathcal{PC}(I;\mathcal{U})\colon\exists\lambda>1\text{ such that }\lambda\widetilde{\mu}\in\Upsilon_\beta(\widetilde{\mu})\big\}$$

is bounded. According to the conditions of Theorem 2, it is not difficult to see

$$\begin{aligned}\big\|\widetilde{\mu}(\sigma)\big\|\leqslant{}&\big\|\widetilde{S}(\sigma)\big(\widetilde{\mu}_0-\mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big)\big\|+\big\|\mathcal{F}\big(t,\widetilde{\mu}(t)\big)\big\|\\&+\left\|\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota\right\|\\&+\left\|\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\right\|\\&+\left\|\int_0^{\sigma}(\sigma-\iota)^{\varsigma-1}\delta(t-\iota)\mathcal{B}\Bigg(\mathcal{B}^*\widetilde{S}^*(T-s)P\big(\beta,\Upsilon_0^T\big)\right.\\&\quad\times\Bigg[\widetilde{\mu}_1-\widetilde{S}(T)\big(\widetilde{\mu}_0-\mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big)-\mathcal{F}\big(T,\widetilde{\mu}(T)\big)\\&\quad-\int_0^{T}(T-\iota)^{\varsigma-1}\delta(T-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota\\&\quad-\int_0^{T}(T-\iota)^{\varsigma-1}\delta(T-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota\\&\quad\left.-\int_0^{T}(T-\iota)^{\varsigma-1}\delta(T-\iota)J(\iota)\,\mathrm{d}\iota-\sum_{0<t_i<T}\widetilde{S}(T-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}(\sigma_i)\big)\Bigg]\,\mathrm{d}\iota\right\|\end{aligned}$$

$$
\begin{aligned}
&+\left\|\int_0^\sigma(\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J(\iota)\,\mathrm{d}\iota\right\|+\left\|\sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i(\widetilde{\mu}(\sigma_i))\right\|\\
&\leqslant \mathcal{M}\|\widetilde{\mu}_0\|+\mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big)+\mathcal{M}M_{\mathcal{F}}\big(1+\big\|\widetilde{\mu}(\sigma)\big\|\big)\\
&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\big\|\widetilde{\mu}(\sigma)\big\|T^{1+\varsigma}}{\varsigma}\right)\\
&\quad+\frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\right.\\
&\quad\left.+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_2}\|\widetilde{\mu}(\sigma)\big\|T^{1+\varsigma}+\|\widetilde{\mu}(\sigma)\|T^{\varsigma}}{\varsigma}\right)\\
&\quad+\frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\left(\|\widetilde{\mu}_1\|+\mathcal{M}\|\widetilde{\mu}_0\|+\mathcal{M}M_{\mathcal{F}}\big(1+\|\widetilde{\mu}_0\|\big)+\mathcal{M}M_{\mathcal{F}}\big(1+\big\|\widetilde{\mu}(T)\big\|\big)\right.\\
&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_1}\|\widetilde{\mu}(T)\|T^{1+\varsigma}}{\varsigma}\right)\\
&\quad+\frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}\right.\\
&\quad\left.+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}+\frac{\mathcal{L}_{\mathcal{R}_2}\big\|\widetilde{\mu}(T)\big\|T^{1+\varsigma}+\big\|\widetilde{\mu}(T)\big\|T^{\varsigma}}{\varsigma}\right)\\
&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\frac{1}{(2\varsigma-1)^{1/2}}T^{(2\varsigma-1)/2}\|\mathcal{B}\|\|\theta\|_{\mathcal{L}^2(I,\mathcal{W})}\\
&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\|\widetilde{\mu}(T)\|\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)}+\mathcal{M}\sum_{i=1}^{m}d_i\Bigg)\\
&\quad+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-\alpha}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\frac{C_J\|\widetilde{\mu}(\sigma)\|\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)}+\mathcal{M}\sum_{i=1}^{m}d_i\\
&\leqslant \mathcal{C}_1+\mathcal{C}_2\|\widetilde{\mu}\|,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{C}_1=&\left(1+\frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\right)\left(\mathcal{M}\|\widetilde{\mu}_0\|+\mathcal{M}M_{\mathcal{F}}(1+\|\widetilde{\mu}_0\|)+\mathcal{M}M_{\mathcal{F}}\right.\\
&+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left(\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_1\|_{L^{1/\lambda}(0,R^+)}\right)\\
&+\frac{\varsigma\mathcal{M}\mathcal{L}_\varphi}{\Gamma(1+\varsigma)}\left(\left[\frac{1-p}{\varsigma-p}\right]^{1-p}T^{\varsigma-p}\|\zeta\|_{L^{1/p}(0,R^+)}+\left[\frac{1-\lambda}{\varsigma-\lambda}\right]^{1-\lambda}T^{\varsigma-\lambda}\|F_2\|_{L^{1/\lambda}(0,R^+)}\right)\\
&\left.+\frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\left[\frac{1-\alpha}{\varsigma-\alpha}\right]^{1-p}T^{\varsigma-\alpha}\|\Theta\|_{L^{1/\alpha}(0,R^+)}+\mathcal{M}\sum_{i=1}^{m}d_i\right)+\frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\|\widetilde{\mu}_1\|
\end{aligned}
$$

and

$$\mathcal{C}_2 = \left(1 + \frac{T^{\varsigma+1}\mathcal{M}\|\mathcal{B}\|^2}{\Gamma(1+\varsigma)}\right)\left(\mathcal{M}M_{\mathcal{F}} + \frac{\varsigma\mathcal{M}}{\Gamma(1+\varsigma)}\frac{\mathcal{L}_{\mathcal{R}_1}T^{1+\varsigma}}{\varsigma} + \frac{\varsigma\mathcal{M}\mathcal{L}_{\varphi}}{\Gamma(1+\varsigma)}\left(\frac{\mathcal{L}_{\mathcal{R}_2}T^{1+\varsigma}+T^{\varsigma}}{\varsigma}\right) + \frac{C_J\mathcal{M}T^{\varsigma}}{\Gamma(1+\varsigma)}\right).$$

Given that $\mathcal{C}_2 < 1$, we have

$$\|\widetilde{\mu}\|_{\mathcal{PC}} \leqslant \frac{\mathcal{C}_1}{1-\mathcal{C}_2}.$$

Thus, the set Ω_β is bounded. By Theorem 1, the operator χ possesses a fixed point, which corresponds to a mild solution of system (1) on the interval I. □

We now proceed to prove the principal theorem of this work.

Theorem 4. *If the corresponding linear control system* (4) *is approximately controllable on I, then system* (1) *is also approximately controllable on I.*

Proof. Using the method applied in Theorem 3, it can be readily shown that for every $\beta > 0$, operator Υ_β possesses a fixed point in $\mathcal{PC}(I,\mathcal{U})$. Hence, for each $\sigma \in I$, there exists a $J_\beta \in \mathcal{N}(\widetilde{\mu}_\beta)$ such that

$$\begin{aligned}\widetilde{\mu}_\beta(\sigma) &= \widetilde{S}(\sigma)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) + \mathcal{F}\big(\sigma,\widetilde{\mu}_\beta(\sigma)\big)\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)(\mathcal{R}_1\widetilde{\mu}_\beta)(\iota)\,\mathrm{d}\iota\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu}_\beta)(\iota),\widetilde{\mu}_\beta(\iota)\big)\,\mathrm{d}\iota\\ &\quad + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)\mathcal{B}\theta_\beta(\iota)\,\mathrm{d}\iota + \int_0^\sigma (\sigma-\iota)^{\varsigma-1}\delta(\sigma-\iota)J_\beta(\iota)\,\mathrm{d}\iota\\ &\quad + \sum_{0<\sigma_i<\sigma}\widetilde{S}(\sigma-\sigma_i)\mathcal{L}_i\big(\widetilde{\mu}_\beta(\sigma_i)\big).\end{aligned}$$

By $\beta P(\beta,\Upsilon_0^T) = I - \Upsilon_0^T P(\beta,\Upsilon_0^T)$, we deduce

$$\widetilde{\mu}_\beta(T) = \widetilde{\mu}_1 - \beta P\big(\beta,\Upsilon_0^T\big)G(\widetilde{\mu}_\beta),$$

where

$$\begin{aligned}G(\widetilde{\mu}_\beta) &= \widetilde{\mu}_1 - \widetilde{S}(T)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) - \mathcal{F}\big(T,\widetilde{\mu}_\beta(T)\big)\\ &\quad - \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)(\mathcal{R}_1\widetilde{\mu}_\beta)(\iota)\,\mathrm{d}\iota\end{aligned}$$

$$
\begin{aligned}
&- \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu}_\beta)(\iota),\widetilde{\mu}_\beta(\iota)\big)\,\mathrm{d}\iota \\
&- \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)J_\beta(\iota)\,\mathrm{d}\iota \\
&- \sum_{0<t_i<T} \widetilde{S}(T-t_i)\mathcal{L}_i\big(\widetilde{\mu}_\beta(t_i)\big).
\end{aligned}
$$

By hypothesis $(\mathrm{H}_{\mathcal{J}})$ and the analysis in Step 5 of Theorem 3, the norm $\|\mathcal{J}(\sigma,x)\|$ is bounded above by $\Theta(\sigma)+C_J\|\widetilde{\mu}_\beta\|$. We denote this bounding function by $\varsigma(\sigma)$. Therefore, we have

$$
\int_0^T \big\|J_\beta(\iota)\big\|\,\mathrm{d}\iota \leqslant \|\varsigma\|_{L^2(I,\mathcal{R}^+)}T^{1/2}.
$$

We thus conclude that the sequence $\{J_\beta\}$ remains uniformly bounded in the space $L^2(I,\mathcal{U})$. Consequently, it possesses at least one weakly convergent subsequence. Without loss of generality, we may retain the notation $\{J_\beta\}$ for this subsequence, which converges weakly to some limit J in $L^2(I,\mathcal{U})$. The function $\mathcal{Q}$ is defined as follows:

$$
\begin{aligned}
\mathcal{Q}(\widetilde{\mu}) = {}& \widetilde{\mu}_1 - \widetilde{S}(T)\big(\widetilde{\mu}_0 - \mathcal{F}\big(0,\widetilde{\mu}(0)\big)\big) - \mathcal{F}\big(T,\widetilde{\mu}(T)\big) \\
&- \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)(\mathcal{R}_1\widetilde{\mu})(\iota)\,\mathrm{d}\iota \\
&- \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)\varphi\big(\iota,(\mathcal{R}_2\widetilde{\mu})(\iota),\widetilde{\mu}(\iota)\big)\,\mathrm{d}\iota \\
&- \int_0^T (T-\iota)^{\varsigma-1}\delta(T-\iota)J(\iota)\,\mathrm{d}\iota - \sum_{0<t_i<T} \widetilde{S}(T-t_i)\mathcal{L}_i\big(\widetilde{\mu}(t_i)\big).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
\big\|\widetilde{\mu}_\beta(T)-\widetilde{\mu}_1\big\| &= \big\|\beta P\big(\beta,\Upsilon_0^T\big)G(\widetilde{\mu}_\beta)\big\| \\
&\leqslant \big\|\beta P\big(\beta,\Upsilon_0^T\big)\mathcal{Q}(\widetilde{\mu})\big\| + \big\|\beta P\big(\beta,\Upsilon_0^T\big)\big[G(\widetilde{\mu}_\beta)-\mathcal{Q}(\widetilde{\mu})\big]\big\| \\
&\leqslant \|\beta P\big(\beta,\Upsilon_0^T\big)\mathcal{Q}\big(\widetilde{\mu}\big)\| + \big\|G(\widetilde{\mu}_\beta)-\mathcal{Q}(\widetilde{\mu})\big\|.
\end{aligned}
$$

As $\beta\to 0$, we have

$$
\big\|\widetilde{\mu}_\beta(T)-\widetilde{\mu}_1\big\| = \big\|\beta P\big(\beta,\Upsilon_0^T\big)G(\widetilde{\mu}_\beta)\big\| \to 0.
$$

Hence, system (1) is approximately controllable on I. Thus, the proof is complete. □

5 Conclusion

This work is concerned with the approximate controllability framework for impulsive neutral functional HIs. The analysis demonstrates the existence of at least one mild solution to the proposed system. We obtain a verifiable sufficient condition for approximate controllability by applying multivalued fixed-point theory and exploiting key properties of the Clarke subdifferential. The present framework can be naturally extended to encompass HIs that incorporate a finite delay.

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