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A priori estimate, existence, and uniqueness of solutions for a third-order differential system with coupled integral boundary conditions

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Abstract. This paper investigates a priori estimate, existence, and uniqueness of solutions for a third-order differential system with coupled integral boundary conditions. The Leray–Schauder continuation principle forms the basis of our approach to establishing the existence result. To demonstrate the applicability of the obtained results, illustrative examples are provided.

Keywords: differential system, coupled integral boundary conditions, a priori estimate of solutions, existence of solutions, uniqueness of solution, Leray–Schauder continuation principle.

1 Introduction

We consider the system of third-order differential equations

$$\begin{aligned} x''' &= f(t,x,y,x',y',x'',y''), \quad t\in(0,1),\\ y''' &= g(t,x,y,x',y',x'',y''), \quad t\in(0,1), \end{aligned} \tag{1}$$

subject to the coupled integral boundary conditions

$$\begin{aligned} &x(0)=0=y(0), \qquad x'(0)=0=y'(0),\\ &x(1)=\int_0^1 y(s)\,\mathrm{d}s, \qquad y(1)=\int_0^1 x(s)\,\mathrm{d}s. \end{aligned} \tag{2}$$

Throughout the paper, we assume that $f,g:[0,1]\times\mathbb{R}^6\to\mathbb{R}$ are continuous. By a solution of (1), (2) we mean a pair $(x,y)\in C^3[0,1]\times C^3[0,1]$ that satisfies system (1) for $0<t<1$ and boundary conditions (2).

The main objective of this paper is to prove existence and uniqueness theorems for the boundary value problem (1), (2). Our approach relies on deriving an a priori estimate

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for a solution and its first and second derivatives and then applying the Leray–Schauder continuation principle [19]. We recall this result below.

Theorem 1. *Let X be a Banach space and $T : X \to X$ be a completely continuous operator (T is continuous and maps bounded sets into relatively compact sets). Suppose that there exists an $r > 0$ such that if $x = \lambda Tx$ for $\lambda \in (0, 1)$, then $\|x\| \leqslant r$. Then T has a fixed point.*

Our main results show that, under suitable growth conditions on the functions f and g, the boundary value problem (1), (2) either admits at least one solution or possesses a unique solution.

The first step in proving the existence result is to reformulate problem (1), (2) as an equivalent system of integral equations. We then introduce an operator on an appropriate function space, reducing the task to showing that this operator admits a fixed point. Finally, the existence of such a fixed point follows from deriving a priori bounds for the solution and for its first and second derivatives.

In order to study the existence of solutions to boundary value problems, it is often necessary to reformulate the problem as an equivalent integral equation and construct the corresponding Green's function. A comprehensive overview of results on Green's functions for stationary problems with nonlocal boundary conditions is given in [16]. Similar problems are also investigated in [14, 15].

Boundary conditions (2) are nonlocal because they indicate solution values not just at the ends of the interval but also within it. Recent achievements in the field of nonlocal problems can be found in papers [2, 17].

The Leray–Schauder continuation principle is a powerful and widely used tool in nonlinear functional analysis, particularly because of its effectiveness in treating boundary value problems for ordinary differential equations. It has been applied, for example, in the works [8, 9, 13], which also serve as motivation for the present study.

Boundary value problems with coupled boundary conditions have received increasing attention in recent decades due to their significant role in many areas of applied mathematics. This provides one of the main motivations for studying problem (1), (2). The existence of solutions to initial value problems for nonlinear first-order differential systems with nonlinear, coupled, nonlocal initial conditions of functional type was investigated in [3]. The existence, multiplicity, and uniqueness of positive solutions to nonlocal boundary value systems with strongly coupled integral boundary conditions were established in [5]. The uniqueness of solutions for differential systems with coupled integral boundary conditions was investigated in [6]. Coupled systems of boundary value problems with nonlocal boundary conditions were examined in [7]. Nonnegative solutions of systems of ordinary differential equations with coupled boundary conditions were studied in [12]. Another motivation is that few papers address problems where the functions in the system depend explicitly on x' and x''. A related contribution is the work [18], which investigates second-order nonlocal boundary value problems with nonlinear terms involving derivatives. The authors in [4] study the case of a third-order equation with a nonlinearity depending on the first-order derivative, under similar nonlocal boundary conditions. The paper [10] provides a fairly general approach to establishing the existence of positive

solutions for systems of boundary value problems with higher order dependence. Further recent developments in the theory of boundary value problems can be found in [1, 11].

The remainder of the paper is organized as follows. Section 2 reformulates the main problem as an equivalent integral system and develops several technical tools needed for our analysis. Section 3 establishes the existence theorem and presents an illustrative example. Section 4 is devoted to the uniqueness result, accompanied by a corresponding example.

2 Preliminaries

Proposition 1. *A pair* (x, y) *is a solution of problem* (1), (2) *if and only if* (x, y) *is a solution of the system of integral equations*

$$\begin{aligned} x(t) &= \int_0^1 G(t,s)f(s)\,\mathrm{d}s + \int_0^1 R(t,s)g(s)\,\mathrm{d}s, \quad t \in [0,1], \\ y(t) &= \int_0^1 G(t,s)g(s)\,\mathrm{d}s + \int_0^1 R(t,s)f(s)\,\mathrm{d}s, \quad t \in [0,1], \end{aligned} \tag{3}$$

where

$$G(t,s) = \begin{cases} \frac{1}{16}t^2(1-s)^2(-s-8) + \frac{1}{2}(t-s)^2, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{1}{16}t^2(1-s)^2(-s-8), & 0 \leqslant t \leqslant s \leqslant 1, \end{cases}$$

$$R(t,s) = \frac{3}{16}t^2\big(-s(1-s)^2\big),$$

$$\begin{aligned} f(s) &\equiv f\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big), \\ g(s) &\equiv g\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big). \end{aligned}$$

By a solution of (3) *we mean a pair* $(x, y) \in C^2[0,1] \times C^2[0,1]$ *that satisfies the integral system* (3) *for* $0 \leqslant t \leqslant 1$.

Proof. If $(x(t), y(t))$ is a solution of problem (1), (2), then $x'''(t) = f(t)$ and $y'''(t) = g(t)$. Integrating these equations thrice and considering that $x(0) = 0$, $y(0) = 0$, $x'(0) = 0$, $y'(0) = 0$, we get

$$x(t) = \frac{1}{2}t^2x''(0) + \frac{1}{2}\int_0^t (t-s)^2 f(s)\,\mathrm{d}s,$$

$$y(t) = \frac{1}{2}t^2y''(0) + \frac{1}{2}\int_0^t (t-s)^2 g(s)\,\mathrm{d}s.$$

In view of $x(1) = \int_0^1 y(s)\,\mathrm{d}s$ and $y(1) = \int_0^1 x(s)\,\mathrm{d}s$, we obtain the system

$$x''(0) - \frac{1}{3}y''(0) = \frac{1}{3}\int_0^1 (1-s)^3 g(s)\,\mathrm{d}s - \int_0^1 (1-s)^2 f(s)\,\mathrm{d}s,$$

$$-\frac{1}{3}x''(0) + y''(0) = \frac{1}{3}\int_0^1 (1-s)^3 f(s)\,\mathrm{d}s - \int_0^1 (1-s)^2 g(s)\,\mathrm{d}s.$$

It follows that

$$x''(0) = \frac{1}{8}\int_0^1 (1-s)^2(-s-8)f(s)\,\mathrm{d}s + \frac{3}{8}\int_0^1 \bigl(-s(1-s)^2\bigr)g(s)\,\mathrm{d}s,$$

$$y''(0) = \frac{3}{8}\int_0^1 \bigl(-s(1-s)^2\bigr)f(s)\,\mathrm{d}s + \frac{1}{8}\int_0^1 (1-s)^2(-s-8)g(s)\,\mathrm{d}s$$

and

$$\begin{aligned} x(t) &= \int_0^t \left(\frac{1}{16}t^2(1-s)^2(-s-8) + \frac{1}{2}(t-s)^2\right)f(s)\,\mathrm{d}s \\ &\quad + \int_t^1 \left(\frac{1}{16}t^2(1-s)^2(-s-8)\right)f(s)\,\mathrm{d}s + \frac{3}{16}t^2\int_0^1 \bigl(-s(1-s)^2\bigr)g(s)\,\mathrm{d}s, \\ y(t) &= \int_0^t \left(\frac{1}{16}t^2(1-s)^2(-s-8) + \frac{1}{2}(t-s)^2\right)g(s)\,\mathrm{d}s \\ &\quad + \int_t^1 \left(\frac{1}{16}t^2(1-s)^2(-s-8)\right)g(s)\,\mathrm{d}s + \frac{3}{16}t^2\int_0^1 \bigl(-s(1-s)^2\bigr)f(s)\,\mathrm{d}s, \end{aligned}$$

and therefore $(x(t), y(t))$ satisfies (3).

If $(x(t), y(t))$ is a solution of integral system (3), then differentiating (3) three times and verifying the continuity, we get that $(x(t), y(t))$ is a solution to problem (1), (2). □

Proposition 2. *If $(x(t), y(t))$ is a solution of boundary value problem* (1), (2)*, then for $t \in [0,1]$,*

$$\bigl|x(t)\bigr| \leqslant \int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{36}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s, \qquad \bigl|y(t)\bigr| \leqslant \frac{1}{36}\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s,$$

$$\bigl|x'(t)\bigr| \leqslant 2\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{18}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s, \qquad \bigl|y'(t)\bigr| \leqslant \frac{1}{18}\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + 2\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s,$$

$$\bigl|x''(t)\bigr| \leqslant 2\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s+\frac{1}{18}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s, \qquad \bigl|y''(t)\bigr| \leqslant \frac{1}{18}\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s+2\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s.$$

Proof. We will show the proof for $|x(t)|$, $|x'(t)|$, and $|x''(t)|$. For $|y(t)|$, $|y'(t)|$, and $|y''(t)|$, the proof is similar. Let us consider

$$\begin{aligned} x(t) &= \int_0^t \left(\frac{1}{16}t^2(1-s)^2(-s-8)+\frac{1}{2}(t-s)^2\right)f(s)\,\mathrm{d}s \\ &\quad + \int_t^1 \left(\frac{1}{16}t^2(1-s)^2(-s-8)\right)f(s)\,\mathrm{d}s + \frac{3}{16}t^2\int_0^1 \bigl(-s(1-s)^2\bigr)g(s)\,\mathrm{d}s. \end{aligned}$$

For all $t \in [0,1]$, we have

$$\begin{aligned} \bigl|x(t)\bigr| &\leqslant \int_0^t \left(\frac{1}{16}t^2(1-s)^2(s+8)+\frac{1}{2}(t-s)^2\right)\bigl|f(s)\bigr|\,\mathrm{d}s \\ &\quad + \int_t^1 \left(\frac{1}{16}t^2(1-s)^2(s+8)\right)\bigl|f(s)\bigr|\,\mathrm{d}s + \frac{3}{16}t^2\int_0^1 s(1-s)^2\bigl|g(s)\bigr|\,\mathrm{d}s \\ &\leqslant \max_{0\leqslant s\leqslant t\leqslant 1}\left\{\frac{1}{16}t^2(1-s)^2(s+8)+\frac{1}{2}(t-s)^2\right\}\int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s \\ &\quad + \max_{0\leqslant t\leqslant s\leqslant 1}\left\{\frac{1}{16}t^2(1-s)^2(s+8)\right\}\int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s \\ &\quad + \max_{0\leqslant t,s\leqslant 1}\left\{\frac{3}{16}t^2s(1-s)^2\right\}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\ &= \int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{6912}{104057+9159\sqrt{129}}\int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{36}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\ &\leqslant \int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s + \int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{36}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\ &= \int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{36}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s. \end{aligned}$$

Consider

$$x'(t) = \int_0^t \left(\frac{1}{8}t(1-s)^2(-s-8) + (t-s)\right) f(s)\,\mathrm{d}s + \int_t^1 \left(\frac{1}{8}t(1-s)^2(-s-8)\right) f(s)\,\mathrm{d}s + \frac{3}{8}t\int_0^1 \left(-s(1-s)^2\right) g(s)\,\mathrm{d}s.$$

For all $t \in [0,1]$, we have

$$\begin{aligned}
\left|x'(t)\right| &\leqslant \int_0^t \left(\frac{1}{8}t(1-s)^2(s+8) + (t-s)\right)\left|f(s)\right|\mathrm{d}s \\
&\quad + \int_t^1 \left(\frac{1}{8}t(1-s)^2(s+8)\right)\left|f(s)\right|\mathrm{d}s + \frac{3}{8}t\int_0^1 s(1-s)^2\left|g(s)\right|\mathrm{d}s \\
&\leqslant \max_{0\leqslant s\leqslant t\leqslant 1}\left\{\frac{1}{8}t(1-s)^2(s+8) + (t-s)\right\}\int_0^t \left|f(s)\right|\mathrm{d}s \\
&\quad + \max_{0\leqslant t\leqslant s\leqslant 1}\left\{\frac{1}{8}t(1-s)^2(s+8)\right\}\int_t^1 \left|f(s)\right|\mathrm{d}s \\
&\quad + \max_{0\leqslant t,s\leqslant 1}\left\{\frac{3}{8}t\,s(1-s)^2\right\}\int_0^1 \left|g(s)\right|\mathrm{d}s \\
&= 2\int_0^t \left|f(s)\right|\mathrm{d}s + \frac{108}{349+85\sqrt{17}}\int_t^1 \left|f(s)\right|\mathrm{d}s + \frac{1}{18}\int_0^1 \left|g(s)\right|\mathrm{d}s \\
&\leqslant 2\int_0^t \left|f(s)\right|\mathrm{d}s + 2\int_t^1 \left|f(s)\right|\mathrm{d}s + \frac{1}{18}\int_0^1 \left|g(s)\right|\mathrm{d}s \\
&= 2\int_0^1 \left|f(s)\right|\mathrm{d}s + \frac{1}{18}\int_0^1 \left|g(s)\right|\mathrm{d}s.
\end{aligned}$$

Consider

$$x''(t) = \int_0^t \left(\frac{1}{8}(1-s)^2(-s-8) + 1\right) f(s)\,\mathrm{d}s + \int_t^1 \left(\frac{1}{8}(1-s)^2(-s-8)\right) f(s)\,\mathrm{d}s + \frac{3}{8}\int_0^1 \left(-s(1-s)^2\right) g(s)\,\mathrm{d}s.$$

For all $t \in [0,1]$, we have

$$\begin{aligned}
\bigl|x''(t)\bigr| &\leqslant \int_0^t \left(\frac{1}{8}(1-s)^2(s+8)+1\right)\bigl|f(s)\bigr|\,\mathrm{d}s \\
&\quad + \int_t^1 \left(\frac{1}{8}(1-s)^2(s+8)\right)\bigl|f(s)\bigr|\,\mathrm{d}s + \frac{3}{8}\int_0^1 s(1-s)^2\bigl|g(s)\bigr|\,\mathrm{d}s \\
&\leqslant \max_{0\leqslant s\leqslant t\leqslant 1}\left\{\frac{1}{8}(1-s)^2(s+8)+1\right\}\int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s \\
&\quad + \max_{0\leqslant t\leqslant s\leqslant 1}\left\{\frac{1}{8}(1-s)^2(s+8)\right\}\int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s \\
&\quad + \max_{0\leqslant s\leqslant 1}\left\{\frac{3}{8}s(1-s)^2\right\}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\
&= 2\int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s + \int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{18}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\
&\leqslant 2\int_0^t \bigl|f(s)\bigr|\,\mathrm{d}s + 2\int_t^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{18}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s \\
&= 2\int_0^1 \bigl|f(s)\bigr|\,\mathrm{d}s + \frac{1}{18}\int_0^1 \bigl|g(s)\bigr|\,\mathrm{d}s. \qquad \square
\end{aligned}$$

3 A priori estimate and existence of solutions

Theorem 2. *Suppose there exist continuous functions $a_i, b_i : [0,1] \to [0,+\infty)$, $i = 1,2,3,4,5,6,7$, such that*

$$\begin{aligned}
&\bigl|f(t,x,y,x',y',x'',y'')\bigr| \\
&\quad \leqslant a_1(t)|x| + a_2(t)|y| + a_3(t)|x'| + a_4(t)|y'| + a_5(t)|x''| + a_6(t)|y''| + a_7(t), \\
&\bigl|g(t,x,y,x',y',x'',y'')\bigr| \\
&\quad \leqslant b_1(t)|x| + b_2(t)|y| + b_3(t)|x'| + b_4(t)|y'| + b_5(t)|x''| + b_6(t)|y''| + b_7(t)
\end{aligned}$$

for $(t,x,y,x',y',x'',y'') \in [0,1]\times\mathbb{R}^6$ and

$$\|d_1\|_1 + \frac{1}{36}\|d_2\|_1 + 2\|d_3\|_1 + \frac{1}{18}\|d_4\|_1 + 2\|d_5\|_1 + \frac{1}{18}\|d_6\|_1 < 1,$$

$$\frac{1}{36}\|d_1\|_1 + \|d_2\|_1 + \frac{1}{18}\|d_3\|_1 + 2\|d_4\|_1 + \frac{1}{18}\|d_5\|_1 + 2\|d_6\|_1 < 1,$$

where $d_i = a_i + b_i$ *and* $\|d\|_1 = \int_0^1 |d(s)|\,\mathrm{d}s$. *Then boundary value problem* (1), (2) *has at least one solution.*

Proof. Let $C^2[0,1]$ denote the Banach space of twice continuously differentiable functions endowed with the norm $\|x\| = \max\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\}$, where $\|u\|_\infty = \max_{t\in[0,1]} |u(t)|$. Thus $E = C^2[0,1] \times C^2[0,1]$ is a Banach space with the norm defined by $\|(x,y)\|_E = \max\{\|x\|, \|y\|\}$. Define an operator T by

$$T(x,y) = \big(T_1(x,y), T_2(x,y)\big), \quad (x,y) \in E,$$

where operators $T_1, T_2 : E \to C^2[0,1]$ are defined by

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 G(t,s) f\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big)\,\mathrm{d}s \\ &\quad + \int_0^1 R(t,s) g\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big)\,\mathrm{d}s, \quad t \in [0,1], \\ T_2(x,y)(t) &= \int_0^1 G(t,s) g\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big)\,\mathrm{d}s \\ &\quad + \int_0^1 R(t,s) f\big(s, x(s), y(s), x'(s), y'(s), x''(s), y''(s)\big)\,\mathrm{d}s, \quad t \in [0,1]. \end{aligned}$$

The existence of a solution to the differential system (1), (2) is therefore equivalent to the existence of a fixed point of T on E. By the Arzelà–Ascoli theorem, the operator T is completely continuous.

Let $\lambda \in (0,1]$ and consider

$$\begin{aligned} x''' &= \lambda f(t,x,y,x',y',x'',y''), \quad t \in (0,1), \\ y''' &= \lambda g(t,x,y,x',y',x'',y''), \quad t \in (0,1), \end{aligned} \tag{4}$$

subject to boundary conditions (2). Our aim is to show that the set of all possible solutions of (4), (2) is a priori bounded in E by a constant independent of λ. In particular, the case $\lambda = 1$ yields an a priori bound for the solutions of problem (1), (2).

Let $(x(t), y(t))$ be a solution of (4), (2) and consider

$$\begin{aligned} \|x'''\|_1 &= \lambda\big\|f(t,x,y,x',y',x'',y'')\big\|_1 \leqslant \big\|f(t,x,y,x',y',x'',y'')\big\|_1 \\ &\leqslant \|a_1 x\|_1 + \|a_2 y\|_1 + \|a_3 x'\|_1 + \|a_4 y'\|_1 + \|a_5 x''\|_1 + \|a_6 y''\|_1 + \|a_7\|_1 \end{aligned}$$

$$\leqslant \left(\|x'''\|_1 + \frac{1}{36}\|y'''\|_1\right)\|a_1\|_1 + \left(\frac{1}{36}\|x'''\|_1 + \|y'''\|_1\right)\|a_2\|_1$$
$$+ \left(2\|x'''\|_1 + \frac{1}{18}\|y'''\|_1\right)\|a_3\|_1 + \left(\frac{1}{18}\|x'''\|_1 + 2\|y'''\|_1\right)\|a_4\|_1$$
$$+ \left(2\|x'''\|_1 + \frac{1}{18}\|y'''\|_1\right)\|a_5\|_1 + \left(\frac{1}{18}\|x'''\|_1 + 2\|y'''\|_1\right)\|a_6\|_1 + \|a_7\|_1.$$

Hence,

$$\left(1 - \|a_1\|_1 - \frac{1}{36}\|a_2\|_1 - 2\|a_3\|_1 - \frac{1}{18}\|a_4\|_1 - 2\|a_5\|_1 - \frac{1}{18}\|a_6\|_1\right)\|x'''\|_1$$
$$+ \left(-\frac{1}{36}\|a_1\|_1 - \|a_2\|_1 - \frac{1}{18}\|a_3\|_1 - 2\|a_4\|_1 - \frac{1}{18}\|a_5\|_1 - 2\|a_6\|_1\right)\|y'''\|_1$$
$$\leqslant \|a_7\|_1.$$

Similarly, we obtain

$$\left(-\|b_1\|_1 - \frac{1}{36}\|b_2\|_1 - 2\|b_3\|_1 - \frac{1}{18}\|b_4\|_1 - 2\|b_5\|_1 - \frac{1}{18}\|b_6\|_1\right)\|x'''\|_1$$
$$+ \left(1 - \frac{1}{36}\|b_1\|_1 - \|b_2\|_1 - \frac{1}{18}\|b_3\|_1 - 2\|b_4\|_1 - \frac{1}{18}\|b_5\|_1 - 2\|b_6\|_1\right)\|y'''\|_1$$
$$\leqslant \|b_7\|_1.$$

Adding the last two inequalities, we obtain

$$\left(1 - \|d_1\|_1 - \frac{1}{36}\|d_2\|_1 - 2\|d_3\|_1 - \frac{1}{18}\|d_4\|_1 - 2\|d_5\|_1 - \frac{1}{18}\|d_6\|_1\right)\|x'''\|_1$$
$$+ \left(1 - \frac{1}{36}\|d_1\|_1 - \|d_2\|_1 - \frac{1}{18}\|d_3\|_1 - 2\|d_4\|_1 - \frac{1}{18}\|d_5\|_1 - 2\|d_6\|_1\right)\|y'''\|_1$$
$$\leqslant \|d_7\|_1.$$

Hence,

$$\|x'''\|_1 \leqslant \frac{\|d_7\|_1}{1 - \|d_1\|_1 - \frac{1}{36}\|d_2\|_1 - 2\|d_3\|_1 - \frac{1}{18}\|d_4\|_1 - 2\|d_5\|_1 - \frac{1}{18}\|d_6\|_1}$$
$$= r_x$$

and

$$\|y'''\|_1 \leqslant \frac{\|d_7\|_1}{1 - \frac{1}{36}\|d_1\|_1 - \|d_2\|_1 - \frac{1}{18}\|d_3\|_1 - 2\|d_4\|_1 - \frac{1}{18}\|d_5\|_1 - 2\|d_6\|_1}$$
$$= r_y.$$

It follows that for all $t \in [0,1]$,

$$\begin{aligned}
\bigl|x(t)\bigr| &\leqslant \|x'''\|_1 + \frac{1}{36}\|y'''\|_1 \leqslant r_x + \frac{1}{36}r_y,\\
\bigl|y(t)\bigr| &\leqslant \frac{1}{36}\|x'''\|_1 + \|y'''\|_1 \leqslant \frac{1}{36}r_x + r_y,\\
\bigl|x'(t)\bigr| &\leqslant 2\|x'''\|_1 + \frac{1}{18}\|y'''\|_1 \leqslant 2\,r_x + \frac{1}{18}r_y,\\
\bigl|y'(t)\bigr| &\leqslant \frac{1}{18}\|x'''\|_1 + 2\|y'''\|_1 \leqslant \frac{1}{18}r_x + 2\,r_y,\\
\bigl|x''(t)\bigr| &\leqslant 2\|x'''\|_1 + \frac{1}{18}\|y'''\|_1 \leqslant 2\,r_x + \frac{1}{18}r_y,\\
\bigl|y''(t)\bigr| &\leqslant \frac{1}{18}\|x'''\|_1 + 2\|y'''\|_1 \leqslant \frac{1}{18}r_x + 2\,r_y.
\end{aligned}$$

According to these estimates, the set of all possible solutions of (4), (2) is a priori bounded in E or

$$\begin{aligned}
\|x\| &= \max\bigl\{\|x\|_\infty, \|x'\|_\infty, \|x''\|_\infty\bigr\} = 2r_x + \frac{1}{18}r_y,\\
\|y\| &= \max\bigl\{\|y\|_\infty, \|y'\|_\infty, \|y''\|_\infty\bigr\} = \frac{1}{18}r_x + 2r_y,\\
\bigl\|(x,y)\bigr\|_E &= \max\bigl\{\|x\|, \|y\|\bigr\} \leqslant 2r_x + 2r_y.
\end{aligned}$$

Therefore, by Theorem 1, T has a fixed point. □

Example 1. Consider boundary value problem for the differential system

$$\begin{aligned}
x''' &= t(1-t)\frac{x^3\cos x}{1+x^2} + \frac{1}{4}\bigl(\sqrt{1+(y'')^2} - 1\bigr) + t^4, \quad t \in (0,1),\\
y''' &= (1-t)^2 y\sin y + \frac{1}{3}(x'' - 2\arctan x'') - 34, \qquad t \in (0,1),
\end{aligned} \tag{5}$$

with boundary conditions (2).

The function

$$f(t,x,y'') = t(1-t)\frac{x^3\cos x}{1+x^2} + \frac{1}{4}\bigl(\sqrt{1+(y'')^2} - 1\bigr) + t^4$$

is continuous for $(t,x,y'') \in [0,1]\times\mathbb{R}^2$, and the function

$$g(t,y,x'') = (1-t)^2 y\sin y + \frac{1}{3}(x'' - 2\arctan x'') - 34$$

is continuous for $(t,y,x'') \in [0,1]\times\mathbb{R}^2$.

Consider

$$\begin{aligned}
\bigl|f(t,x,y'')\bigr| &\leqslant t(1-t)\frac{x^2|\cos x|}{1+x^2}|x| + \frac{1}{4}\left|\frac{\sqrt{1+(y'')^2}-1}{y''}\right||y''| + t^4\\
&\leqslant t(1-t)|x| + \frac{1}{4}|y''| + t^4,
\end{aligned}$$

$$\begin{aligned}\bigl|g(t,y,x'')\bigr| &\leqslant (1-t)^2|y||\sin y| + \frac{1}{3}\left|\frac{x''-2\arctan x''}{x''}\right||x''| + 34\\ &\leqslant (1-t)^2|y| + \frac{1}{3}|x''| + 34.\end{aligned}$$

We have

$$\begin{aligned}&a_1 = t(1-t), \quad a_2 = 0, \quad a_3 = 0, \quad a_4 = 0, \quad a_5 = 0, \quad a_6 = \frac{1}{4}, \quad a_7 = t^4,\\ &b_1 = 0, \quad b_2 = (1-t)^2, \quad b_3 = 0, \quad b_4 = 0, \quad b_5 = \frac{1}{3}, \quad b_6 = 0, \quad b_7 = 34,\\ &d_1 = t(1-t), \quad d_2 = (1-t)^2, \quad d_3 = 0, \quad d_4 = 0, \quad d_5 = \frac{1}{3}, \quad d_6 = \frac{1}{4}, \quad d_7 = t^4+34,\\ &\|d_1\|_1 + \frac{1}{36}\|d_2\|_1 + 2\|d_3\|_1 + \frac{1}{18}\|d_4\|_1 + 2\|d_5\|_1 + \frac{1}{18}\|d_6\|_1 = \frac{185}{216} < 1,\\ &\frac{1}{36}\|d_1\|_1 + \|d_2\|_1 + \frac{1}{18}\|d_3\|_1 + 2\|d_4\|_1 + \frac{1}{18}\|d_5\|_1 + 2\|d_6\|_1 = \frac{185}{216} < 1.\end{aligned}$$

Thus, by Theorem 2, problem (5), (2) has at least one (nontrivial) solution. Since $r_x = r_y = 36936/155$, we have the following estimates:

$$\begin{aligned}&\bigl|x(t)\bigr| \leqslant \frac{37962}{155}, \qquad \bigl|x'(t)\bigr| \leqslant \frac{75924}{155}, \qquad \bigl|x''(t)\bigr| \leqslant \frac{75924}{155},\\ &\bigl|y(t)\bigr| \leqslant \frac{37962}{155}, \qquad \bigl|y'(t)\bigr| \leqslant \frac{75924}{155}, \qquad \bigl|y''(t)\bigr| \leqslant \frac{75924}{155}.\end{aligned}$$

4 Uniqueness of solution

Theorem 3. *Suppose there exist continuous functions* $\alpha_i,\ \beta_i : [0,1] \to [0,+\infty)$, $i = 1,2,3,4,5,6$, *such that*

$$\begin{aligned}&\bigl|f(t,x_1,y_1,x_1',y_1',x_1'',y_1'') - f(t,x_2,y_2,x_2',y_2',x_2'',y_2'')\bigr|\\ &\quad\leqslant \alpha_1(t)|x_1-x_2| + \alpha_2(t)|y_1-y_2| + \alpha_3(t)|x_1'-x_2'|\\ &\qquad + \alpha_4(t)|y_1'-y_2'| + \alpha_5(t)|x_1''-x_2''| + \alpha_6(t)|y_1''-y_2''|\end{aligned}$$

and

$$\begin{aligned}&\bigl|g(t,x_1,y_1,x_1',y_1',x_1'',y_1'') - g(t,x_2,y_2,x_2',y_2',x_2'',y_2'')\bigr|\\ &\quad\leqslant \beta_1(t)|x_1-x_2| + \beta_2(t)|y_1-y_2| + \beta_3(t)|x_1'-x_2'|\\ &\qquad + \beta_4(t)|y_1'-y_2'| + \beta_5(t)|x_1''-x_2''| + \beta_6(t)|y_1''-y_2''|\end{aligned}$$

for $(t,x_1,y_1,x_1',y_1',x_1'',y_1''), (t,x_2,y_2,x_2',y_2',x_2'',y_2'') \in [0,1]\times\mathbb{R}^6$.

Then boundary value problem (1), (2) *has exactly one solution provided that*

$$\begin{aligned}&\|\gamma_1\|_1 + \frac{1}{36}\|\gamma_2\|_1 + 2\|\gamma_3\|_1 + \frac{1}{18}\|\gamma_4\|_1 + 2\|\gamma_5\|_1 + \frac{1}{18}\|\gamma_6\|_1 < 1,\\ &\frac{1}{36}\|\gamma_1\|_1 + \|\gamma_2\|_1 + \frac{1}{18}\|\gamma_3\|_1 + 2\|\gamma_4\|_1 + \frac{1}{18}\|\gamma_5\|_1 + 2\|\gamma_6\|_1 < 1,\end{aligned}$$

where $\gamma_i = \alpha_i + \beta_i$.

Proof. If $x_2 = y_2 = x_2' = y_2' = x_2'' = y_2'' = 0$, we get

$$\begin{aligned}
&\big|f(t,x_1,y_1,x_1',y_1',x_1'',y_1'')\big| \\
&\quad\leqslant \alpha_1(t)|x_1| + \alpha_2(t)|y_1| + \alpha_3(t)|x_1'| + \alpha_4(t)|y_1'| + \alpha_5(t)|x_1''| + \alpha_6(t)|y_1''| \\
&\qquad + \big|f(t,0,0,0,0,0,0)\big|, \\
&\big|g(t,x_1,y_1,x_1',y_1',x_1'',y_1'')\big| \\
&\quad\leqslant \beta_1(t)|x_1| + \beta_2(t)|y_1| + \beta_3(t)|x_1'| + \beta_4(t)|y_1'| + \beta_5(t)|x_1''| + \beta_6(t)|y_1''| \\
&\qquad + \big|g(t,0,0,0,0,0,0)\big|,
\end{aligned}$$

for $(t,x_1,y_1,x_1',y_1',x_1'',y_1'') \in [0,1]\times\mathbb{R}^6$. Hence, in view of Theorem 2, boundary value problem (1), (2) has at least one solution. To prove the uniqueness of a solution, let $(x_1(t),y_1(t))$, $(x_2(t),y_2(t))$ be two solutions for (1), (2). If $u(t) = x_1(t) - x_2(t)$ and $v(t) = y_1(t) - y_2(t)$, we get

$$\begin{aligned}
u'''(t) &= f\big(t,x_1(t),y_1(t),x_1'(t),y_1'(t),x_1''(t),y_1''(t)\big) \\
&\quad - f\big(t,x_2(t),y_2(t),x_2'(t),y_2'(t),x_2''(t),y_2''(t)\big), \\
v'''(t) &= g\big(t,x_1(t),y_1(t),x_1'(t),y_1'(t),x_1''(t),y_1''(t)\big) \\
&\quad - g\big(t,x_2(t),y_2(t),x_2'(t),y_2'(t),x_2''(t),y_2''(t)\big), \\
u(0) &= 0 = v(0), \qquad u'(0) = 0 = v'(0), \\
u(1) &= \int_0^1 v(s)\,\mathrm{d}s, \qquad v(1) = \int_0^1 u(s)\,\mathrm{d}s.
\end{aligned}$$

Let us consider

$$\begin{aligned}
\|u'''\|_1 &= \big\|f(t,x_1,y_1,x_1',y_1',x_1'',y_1'') - f(t,x_2,y_2,x_2',y_2',x_2'',y_2'')\big\|_1 \\
&\leqslant \|\alpha_1 u\|_1 + \|\alpha_2 v\|_1 + \|\alpha_3 u'\|_1 + \|\alpha_4 v'\|_1 + \|\alpha_5 u''\|_1 + \|\alpha_6 v''\|_1 \\
&\leqslant \left(\|u'''\|_1 + \frac{1}{36}\|v'''\|_1\right)\|\alpha_1\|_1 + \left(\frac{1}{36}\|u'''\|_1 + \|v'''\|_1\right)\|\alpha_2\|_1 \\
&\quad + \left(2\|u'''\|_1 + \frac{1}{18}\|v'''\|_1\right)\|\alpha_3\|_1 + \left(\frac{1}{18}\|u'''\|_1 + 2\|v'''\|_1\right)\|\alpha_4\|_1 \\
&\quad + \left(2\|u'''\|_1 + \frac{1}{18}\|v'''\|_1\right)\|\alpha_5\|_1 + \left(\frac{1}{18}\|u'''\|_1 + 2\|v'''\|_1\right)\|\alpha_6\|_1 \\
&= \left(\|\alpha_1\|_1 + \frac{1}{36}\|\alpha_2\|_1 + 2\|\alpha_3\|_1 + \frac{1}{18}\|\alpha_4\|_1 + 2\|\alpha_5\|_1 + \frac{1}{18}\|\alpha_6\|_1\right)\|u'''\|_1 \\
&\quad + \left(\frac{1}{36}\|\alpha_1\|_1 + \|\alpha_2\|_1 + \frac{1}{18}\|\alpha_3\|_1 + 2\|\alpha_4\|_1 + \frac{1}{18}\|\alpha_5\|_1 + 2\|\alpha_6\|_1\right)\|v'''\|_1.
\end{aligned}$$

Similarly, we obtain

$$\|v'''\|_1 \leqslant \left(\|\beta_1\|_1 + \frac{1}{36}\|\beta_2\|_1 + 2\|\beta_3\|_1 + \frac{1}{18}\|\beta_4\|_1 + 2\|\beta_5\|_1 + \frac{1}{18}\|\beta_6\|_1\right)\|u'''\|_1 + \left(\frac{1}{36}\|\beta_1\|_1 + \|\beta_2\|_1 + \frac{1}{18}\|\beta_3\|_1 + 2\|\beta_4\|_1 + \frac{1}{18}\|\beta_5\|_1 + 2\|\beta_6\|_1\right)\|v'''\|_1.$$

Adding the last two inequalities, we obtain

$$\|u'''\|_1 + \|v'''\|_1 \leqslant \left(\|\gamma_1\|_1 + \frac{1}{36}\|\gamma_2\|_1 + 2\|\gamma_3\|_1 + \frac{1}{18}\|\gamma_4\|_1 + 2\|\gamma_5\|_1 + \frac{1}{18}\|\gamma_6\|_1\right)\|u'''\|_1 + \left(\frac{1}{36}\|\gamma_1\|_1 + \|\gamma_2\|_1 + \frac{1}{18}\|\gamma_3\|_1 + 2\|\gamma_4\|_1 + \frac{1}{18}\|\gamma_5\|_1 + 2\|\gamma_6\|_1\right)\|v'''\|_1.$$

It follows that $\|u'''\|_1 = 0$, $\|v'''\|_1 = 0$, and hence $u(t) = 0$, $v(t) = 0$ for every $t \in [0,1]$. We get that $(x_1(t), y_1(t)) = (x_2(t), y_2(t))$ for every $t \in [0,1]$. □

Example 2. Let us now consider boundary value problem for the differential system

$$\begin{aligned} x''' &= \frac{t^2(y'')^2}{(y'')^2+1} + 3, \quad t \in (0,1), \\ y''' &= t^3 \sin x'' - 5, \quad t \in (0,1), \end{aligned} \tag{6}$$

with boundary conditions (2).

The function $f(t, y'') = t^2(y'')^2/((y'')^2+1)+3$ is continuous for $(t, y'') \in [0,1]\times\mathbb{R}$, and the function $g(t, x'') = t^3 \sin x'' - 5$ is continuous for $(t, x'') \in [0,1] \times \mathbb{R}$.

Consider

$$\begin{aligned} \left|f(t, y_1'') - f(t, y_2'')\right| &= \left|\frac{t^2(y_1'')^2}{(y_1'')^2+1} - \frac{t^2(y_2'')^2}{(y_2'')^2+1}\right| = t^2\left|\frac{(y_1''-y_2'')(y_1''+y_2'')}{((y_1'')^2+1)((y_2'')^2+1)}\right| \\ &= t^2\left|y_1'' - y_2''\right|\left|\frac{y_1''}{((y_1'')^2+1)((y_2'')^2+1)} + \frac{y_2''}{((y_1'')^2+1)((y_2'')^2+1)}\right| \\ &\leqslant t^2|y_1'' - y_2''|, \\ \left|g(t, x_1'') - g(t, x_2'')\right| &= \left|t^3 \sin x_1'' - t^3 \sin x_2''\right| = t^3|\sin x_1'' - \sin x_2''| \leqslant t^3|x_1'' - x_2''|. \end{aligned}$$

We have

$$\begin{gathered} \alpha_1 = 0, \quad \alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0, \quad \alpha_5 = 0, \quad \alpha_6 = t^2, \\ \beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 0, \quad \beta_5 = t^3, \quad \beta_6 = 0, \\ \gamma_1 = 0, \quad \gamma_2 = 0, \quad \gamma_3 = 0, \quad \gamma_4 = 0, \quad \gamma_5 = t^3, \quad \gamma_6 = t^2, \\ \|\gamma_1\|_1 + \frac{1}{36}\|\gamma_2\|_1 + 2\|\gamma_3\|_1 + \frac{1}{18}\|\gamma_4\|_1 + 2\|\gamma_5\|_1 + \frac{1}{18}\|\gamma_6\|_1 = \frac{14}{27} < 1, \\ \frac{1}{36}\|\gamma_1\|_1 + \|\gamma_2\|_1 + \frac{1}{18}\|\gamma_3\|_1 + 2\|\gamma_4\|_1 + \frac{1}{18}\|\gamma_5\|_1 + 2\|\gamma_6\|_1 = \frac{49}{72} < 1. \end{gathered}$$

Thus, by Theorem 3, problem (6), (2) has exactly one (nontrivial) solution.

Conflicts of interest. The author declares no conflicts of interest.

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