Nonlinear Analysis: Modelling and Control, Vol. 31, No. 6, 1385–1403
https://doi.org/10.15388/namc.2026.31.49073

Vilnius University Press

On the stability of difference schemes for elliptic nonlocal boundary value problems

Allaberen Ashyralyev[a,b,c,1], **Charyyar Ashyralyyev**[a,b,c,2], **Makhmud A. Sadybekov**[b]

[a]Department of Mathematics,
Faculty of Engineering and Natural Sciences,
Bahcesehir University,
Beşiktaş, Istanbul, 34353, Türkiye
aallaberen@gmail.com; charyar@gmail.com

[b]Institute of Mathematics and Mathematical Modeling,
Almaty, 050010, Kazakhstan
sadybekov@math.kz

[c]National University of Uzbekistan named after Mirzo Ulugbek,
Tashkent, 100174, Uzbekistan

Received: May 20, 2026 / **Revised:** August 25, 2026 / **Published online:** October 15, 2026

Abstract. Nonlocal boundary value problems for elliptic equations arise in many applied models; however, their numerical analysis in Banach spaces, particularly under the Samarskii–Ionkin condition, remains insufficiently studied. Motivated by this gap, the present work develops first- and second-order two-step difference schemes for the numerical solution of nonlocal boundary value problems associated with elliptic equations involving a positive definite operator in a Banach space. Coercive stability estimates for the proposed schemes are established. Furthermore, the abstract results are applied to the construction of stable difference schemes for multidimensional elliptic partial differential equations. Numerical experiments for two- and three-dimensional problems confirm the efficiency and applicability of the proposed method.

Keywords: nonlocal boundary value problem, elliptic equation, stability estimate, coercive stability, difference scheme, Banach space, Samarskii–Ionkin condition.

1 Introduction

Elliptic equations (EEs) with nonlocal boundary conditions arise in various applied models where boundary behavior depends on global system properties. In particular, Samarskii–Ionkin-type conditions lead to operator-differential formulations in Banach spaces and pose substantial analytical and numerical challenges. Classical discretization

[1,2] The author was financially supported by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan (grant No. AP23487362).

© 2026 The Author(s). Published by Vilnius University Press
This is an Open Access article distributed under the terms of the Creative Commons Attribution Licence, which permits unrestricted use, distribution, and reproduction in any medium, provided the original author and source are credited.

techniques are often insufficient to ensure stability and robustness for such problems, especially in the presence of positive definite operators.

The construction of stable and accurate difference schemes (DSs) for EEs with nonlocal boundary conditions therefore remains an important open issue. Of particular interest are schemes that preserve coercive stability at the discrete level, as such properties are essential for well-posedness and reliable numerical approximation.

Motivated by these considerations, this work develops first- and second-order two-step DSs for EEs subject to the Samarskii–Ionkin condition. Coercive stability estimates (SEs) are established in an abstract Banach space setting and then applied to the numerical solution of multidimensional elliptic problems, with computational experiments confirming the theoretical results.

Elliptic partial differential equations (PDEs) are very common in mathematics and are fundamental to a wide range of physical and engineering problems. The well-posedness of the local boundary value problem (BVP) for the elliptic equation

$$-\vartheta''(t) + A\vartheta(t) = f(t), \quad 0 < t < T, \qquad \vartheta(0) = \vartheta_0, \quad \vartheta(T) = \vartheta_T \tag{1}$$

in an unspecified Banach space E with a positive operator (PO) A, as well as its applications, has been thoroughly investigated in the literature; see, for instance, [7, 24, 29].

In the mathematical modeling of real processes, EEs are usually coupled with local boundary conditions. However, in many applications, these conditions are inadequate, and nonlocal boundary conditions are required to accurately describe physical, biological, chemical, and environmental processes. Such conditions arise when boundary data depend on the solution inside the domain or cannot be measured directly.

Extensive research has been devoted to examining the well-posedness of nonlocal boundary value problems (NBVPs) arising in partial differential and difference equations. In recent decades, significant attention has been devoted to the theoretical and numerical study of NBVPs for elliptic and parabolic equations. Well-posedness results for Bitsadze–Samarskii-type NBVPs in Banach spaces were established in [1,8,25], and convergence of DSs for delay parabolic equations was analyzed in [2]. Existence and asymptotic behavior of constrained minimizers for EEs involving two nonlocal terms were investigated in [35]. A stable difference scheme for the identification problem related to the elliptic telegraph equation was proposed and analyzed in [3], where the absolute stability and convergence of the method were established.

The numerical treatment of elliptic type problems with nonlocal boundary conditions was discussed in [8, 10], providing efficient computational approaches and error analyses.

An approximation method for the inverse problem for EE problem with mixed boundary and overdetermination conditions was developed in [10], ensuring numerical accuracy and stability of the reconstructed solution.

The finite DSs for the overdetermined problem for EE with mixed boundary conditions were presented in [9], where convergence and consistency of the scheme were demonstrated.

A fourth-order accurate DS for the overdetermined elliptic problem was constructed and theoretically justified in [9], contributing to the development of high-precision

methods for nonlocal elliptic models. Inverse problems for diffusion and Laplace equations with nonlocal or fractional boundary conditions were treated in [20], while well-posedness of difference and differential elliptic problems with nonlocal conditions was further addressed in [19]. The solvability of essentially nonlinear EEs subject to nonlocal boundary conditions was examined in [31], where rigorous sufficient conditions ensuring the existence of solutions were derived. Numerical and spectral methods for EEs with integral boundary conditions were investigated in [28], and nonclassical elliptic systems with integral conditions were analyzed in [11]. Abstract elliptic problems with integral boundary conditions were studied in [7].

PhD theses [15, 25, 32, 33] form an important body of work devoted to the development of functional-analytic frameworks and finite-difference techniques for NBVPs and fractional spaces associated with positive operators under nonlocal or periodic conditions.

In particular, [33] focuses on constructing high-order accurate DSs for Bitsadze–Samarskii problems, while [15] and [32] study fractional spaces generated by POs under various nonlocal conditions. The work [25] addresses finite-difference and analytical approaches for elliptic problems with nonlocal boundary conditions, emphasizing stability and solvability aspects.

NBVPs, finite DSs, and asymptotic properties of differential operators have been actively studied in recent years; see, for example, Būda et al. [13], Bakhit et al. [12].

In [17], Ionkin investigated a nonlocal problem for a one-dimensional parabolic equation arising in models of nonclassical heat conduction, proving existence of solutions; their stability was later established in [18]. Subsequently, Samarskii [27] proposed a nonlocal boundary value problem formulation for parabolic equations in one space variable that unifies classical initial BVPs with Ionkin's problem.

Resently, various NBVPs with Samarskii–Ionkin condition for PDEs have been investigated by many researchers. A generalized Samarskii–Ionkin-type problem for the Poisson equation was formulated and analyzed in [14], where existence and uniqueness results were obtained for spherical domains. The solvability of BVPs for the biharmonic equation with periodic conditions was investigated in [30]. Spectral problems for EEs with nonlocal boundary conditions were examined in [21]. A generalization of the Robin problem for the Laplace equation was proposed and analyzed in [34]. The spectral properties and expansions for nonself-adjoint differential operators were discussed in [16]. BVPs of the Samarskii–Ionkin type for the Laplace operator in a ball were considered in [26], while nonlocal problems with generalized Samarskii–Ionkin conditions for nonstationary, quasiparabolic, and pseudoparabolic equations were developed in [22, 23].

In a Banach space E, the NBVP for the EE with Samarskii–Ionkin condition

$$\begin{aligned} &-\frac{\mathrm{d}^2}{\mathrm{d}t^2}v(t) + Av(t) = f(t), \quad 0 < t < T, \\ &v(0) = v(T) + \phi, \qquad v'(T) + \mu v(T) = \psi \end{aligned} \tag{2}$$

with strongly PO A and $\mu \geqslant 0$ was investigated in [5]. The theorems on the well-posedness of this NBVP were established. In the applications, the new coercive SEs for solution of three BVPs for EEs with Samarskii–Ionkin condition were obtained.

In this paper, we construct and analyze first- and second-order accurate DSs for the approximate solution of NBVP (2). The first-order accurate DS is given by

$$\begin{aligned} &-\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+Av_k=f_k, \quad f_k=f(t_k), \quad t_k\in[0,T]_\tau, \\ &v_0=v_N+\phi, \qquad (1+\mu\tau)v_N-v_{N-1}=\tau\psi, \end{aligned} \tag{3}$$

while the second-order DS takes the form

$$\begin{aligned} &-\frac{v_{k+1}-2v_k+v_{k-1}}{\tau^2}+Av_k=f_k, \quad t_k\in[0,T]_\tau, \\ &v_0=v_N+\phi, \qquad (3+2\tau\mu)v_N-4v_{N-1}+v_{N-2}=2\tau\psi. \end{aligned} \tag{4}$$

Here N is a given natural number, and $[0,T]_\tau=\{t_k=k\tau,\ k=\overline{1,N},\ N\tau=T\}$ denotes the uniform grid on $[0,T]$ with a sufficiently small step size $\tau>0$.

The well-posedness of the DSs (3) and (4) in Banach spaces is rigorously established. In applications, the coercive SEs in the maximum norm (with respect to one variable) are derived for the solutions of the DSs arising in the numerical treatment of three elliptic problems. The proposed approach is implemented and validated on two- and three-dimensional EE problems.

We now recall several auxiliary results from [7] that will be used in the subsequent analysis. Accordingly, consider the following second-order accurate DS for BVP (1):

$$\begin{aligned} &-\frac{\vartheta_{k+1}-2\,\vartheta_k+\vartheta_{k-1}}{\tau^2}+A\vartheta_k=f_k, \quad t_k\in[0,T]_\tau, \\ &\vartheta_0 \text{ and } \vartheta_N \text{ are given elements of } E \end{aligned} \tag{5}$$

This DS admits a unique solution, and the following representation holds:

$$\begin{aligned} \vartheta_k=&\left(I-R^{2N}\right)^{-1}\Bigg\{\left(R^k-R^{2N-k}\right)\vartheta_0+\left(R^{N-k}-R^{N+k}\right)\vartheta_N \\ &-\left(R^{N-k}-R^{N+k}\right)C\sum_{i=1}^{N-1}\left(R^{N-i}-R^{N+i}\right)f_i\tau\Bigg\} \\ &+C\sum_{i=1}^{N-1}\left(R^{|k-i|}-R^{k+i}\right)f_i\tau, \quad 1\leqslant k\leqslant N-1, \\ F=F(\tau,A)=&\frac{\tau A}{2}+\sqrt{\left(\frac{\tau A}{2}\right)^2+A}, \\ R=&\left(I+\tau F(\tau,A)\right)^{-1}, \qquad C=(I+\tau F)(2I+\tau F)^{-1}F^{-1}. \end{aligned} \tag{6}$$

Observe that $F(\tau,A)\neq A^{1/2}$; however, $F(\tau,A)\to A^{1/2}$ as $\tau\to 0$. Moreover, under appropriate assumptions on A, it possesses the same spectral properties as $A^{1/2}$. Formula (6) indicates that the stability and well-posedness analysis of the DS (5) essentially relies on the behavior of the operator powers of R associated with the general operator A.

Theorem 1. (*See* [7].) *Let A be a strongly PO in the Banach space E, and $A\vartheta_0 - f_1$, $A\vartheta_N - f_{N-1} \in E'_\alpha$. Consequently, the solution of the DS* (5) *in $C^\alpha_\tau(E)$ obeys the coercive SE*

$$\begin{aligned}&\big\|\{\tau^{-2}(\vartheta_{k+1}-2)\vartheta_k+\vartheta_{k-1}\}_1^{N-1}\big\|_{C^\alpha_\tau(E)}+\big\|\{A\vartheta_k\}_1^{N-1}\big\|_{C^\alpha_\tau(E)}\\&\quad+\big\|\{\tau^{-2}(\vartheta_{k+1}-2\vartheta_k+\vartheta_{k-1})\}_1^{N-1}\big\|_{C_\tau(E'_\alpha)}\\&\quad\leqslant M\alpha^{-1}(1-\alpha)^{-1}\|f^\tau\|_{C^\alpha_\tau(E)}+M\big(\|A\vartheta_0-f_1\|_{E'_\alpha}+\|A\vartheta_N-f_{N-1}\|_{E'_\alpha}\big),\end{aligned}$$

where M does not depend on ϑ_0, ϑ_N, α, f^τ, and τ.

Theorem 2. (*See* [7].) *Let A be a strongly PO in the Banach space E, and $A\vartheta_0$, $A\vartheta_N \in E'_\alpha$. Consequently, the solution of the DS* (5) *in $C_\tau(E'_\alpha)$ obeys the coercive SE*

$$\begin{aligned}&\big\|\{\tau^{-2}(\vartheta_{k+1}-2\vartheta_k+\vartheta_{k-1})\}_1^{N-1}\big\|_{C_\tau(E'_\alpha)}+\big\|\{A\vartheta_k\}_1^{N-1}\big\|_{C_\tau(E'_\alpha)}\\&\quad\leqslant M\big(\alpha^{-1}(1-\alpha)^{-1}\|f^\tau\|_{C_\tau(E'_\alpha)}+\|A\vartheta_0\big\|_{E'_\alpha}+\|A\vartheta_N\|_{E'_\alpha}\big).\end{aligned}$$

Here M does not depend on ϑ_0, ϑ_N, α, f^τ, and τ. The Banach space $E'_\alpha = E'_\alpha(F,E)$, $0<\alpha<1$, consists of all $\vartheta \in E$ with finite norm

$$\|\vartheta\|_{E'_\alpha} = \sup_{z>0} z^\alpha \big\|F(zI+F)^{-1}\vartheta\big\|_E + \|\vartheta\|_E.$$

2 Stability estimates

The lemma below will be employed in the analysis that follows.

Lemma 1. (*See* [5].) *Let A be PO in a Banach space E. Then the operator $(I-(F-\mu I)\times(F+\mu I)^{-1}\mathrm{e}^{-TB})$ has an inverse $(I-(F-\mu I)(F+\mu I)^{-1}\mathrm{e}^{-TF})^{-1}$, and the following estimates hold*:

$$\begin{aligned}&\big\|F(F+\mu I)^{-1}\big\|_{E\to E}\leqslant M(F),\qquad \big\|(F-\mu I)(F+\mu I)^{-1}\big\|_{E\to E}\leqslant M(F),\\&\big\|\big(I-(F-\mu I)(F+\mu I)^{-1}\mathrm{e}^{-TF}\big)^{-1}\big\|_{E\to E}\leqslant\frac{M(F)}{1-M(F)\mathrm{e}^{-\delta(F)T}}.\end{aligned}$$

Here $M(F)$ is a positive constant that depends on operator F, and $\delta(F)>\ln M(F)/T$.

Lemma 2. (*See* [5].) *For the small $\tau \geqslant 0$ and the PO A in a Banach space E, the following estimates obey*:

$$\begin{aligned}&\big\|F\big(\mu I+(1+\mu\tau)F\big)^{-1}\big\|_{E\to E}\leqslant M(F),\\&\big\|(F-\mu I)\big(\mu\mathbb{I}+(1+\mu\tau)F\big)^{-1}\big\|_{E\to E}\leqslant M(F),\end{aligned}\tag{7}$$

$$\begin{aligned}&\big\|F\big(2(F+\mu I)+4\mu\tau\mathbb{B}+(3+2\mu\tau)\tau F^2\big)^{-1}\big\|_{E\to E}\leqslant M(F),\\&\big\|F\big(2F-2\mu I-\tau F^2\big)\big(2(F+\mu I)+4\mu\tau F+(3+2\mu\tau)\tau F^2\big)^{-1}\big\|_{E\to E}\leqslant M(F),\end{aligned}\tag{8}$$

$$\big\|\big(I-Q(\tau F)\big)^{-1}\big\|_{E\to E}\leqslant M(F),\qquad \big\|\big(I-P(\tau F)\big)^{-1}\big\|_{E\to E}\leqslant M(F),\tag{9}$$

where

$$Q(\tau F) = (F - \mu I)\big(\mu I + (1 + \mu\tau)F\big)^{-1} R^{N-1},$$
$$P(\tau F) = (2F - 2\mu I - \tau F^2)\big(2(F + \mu I) + 4\mu\tau F + (3 + 2\mu\tau)\tau F^2\big)^{-1} R^{N-2}.$$

Now, let us find solution of DSs (3) and (4). Applying formula (6) and nonlocal condition

$$\upsilon_0 = \upsilon_N + \phi, \tag{10}$$

one can get formula

$$\upsilon_k = \big(I - R^{2N}\big)^{-1} \Bigg\{ \big(R^k - R^{2N-k}\big)\phi + \big(R^k - R^{2N-k} + R^{N-k} - R^{N+k}\big)\upsilon_N$$
$$- \big(R^{N-1-k} - R^{N-1+k}\big)(2I + \tau R)^{-1} F^{-1} \sum_{i=1}^{N-1} \big(R^{N-i} - R^{N+i}\big) f_i \tau \Bigg\}$$
$$+ R^{-1}(2I + \tau F)^{-1} F^{-1} \sum_{i=1}^{N-1} \big(\mathbb{R}^{|k-i|} - R^{k+i}\big) f_i \tau, \quad 1 \leqslant k \leqslant N-1, \tag{11}$$

for the solutions of DSs (3) and (4).

Here υ_N is an unknown element of the Banach space. Using formula (11) together with the nonlocal condition $(1 + \mu\tau)\upsilon_N - \upsilon_{N-1} = \tau\psi$, we obtain

$$\upsilon_N = \big(\mu I + (1 + \mu\tau)F\big)^{-1} \big(I - Q(\tau F)\big)^{-1} \Bigg\{ \big(I - R^N\big)^{-1} \Bigg\{ BR^N(2I + \tau F)\phi$$
$$+ \sum_{i=1}^{N-1} \big(R^{N-1-i} - R^{N-1+i}\big) f_i \tau \Bigg\} + \big(I + R^N\big) R^{-1} \psi \Bigg\}. \tag{12}$$

So, we get formulas (6) and (12) for the solution of the DS (3).

Second, to find the solution of DS (4), we use formula (11) and nonlocal condition $(3 + 2\tau\mu)\upsilon_N - 4\upsilon_{N-1} + \upsilon_{N-2} = 2\tau\psi$. Thus

$$\upsilon_N = \big(2(F + \mu I) + 4\mu\tau F + (3 + 2\mu\tau)\tau F^2\big)^{-1} \Bigg\{ \big(I - P(\tau F)\big)^{-1} \big(I - R^N\big)^{-1}$$
$$\times \Bigg\{ (2I + \tau F)\big(-2 - 2\tau F + \tau^2 F^2\big)\mathbb{B}\mathbb{R}^N \phi + 4 f_{N-1}\tau$$
$$+ \big(R - R^{2N-3}\big) f_{N-1}\tau - 4 \sum_{i=1}^{N-2} \big(R^{N-2-i} - R^{N-2+i}\big) f_i \tau$$
$$+ \big(R^{-1} + R\big) \sum_{i=1}^{N-2} \big(R^{N-2-i} - R^{N-2+i}\big) f_i \tau \Bigg\} + 2\big(I + R^N\big) R^{-2} \psi \Bigg\}. \tag{13}$$

Consequently, the solution of DS (4) is given by formulas (6) and (13).

It should be noted that the coercivity inequality

$$\begin{aligned}&\big\|\{Av_k\}_1^{N-1}\big\|_{C_\tau^\alpha(E)}+\big\|\big\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\big\}_1^{N-1}\big\|_{C_\tau^\alpha(E)}\\&\quad\leqslant M\big[\|f\|_{C_\tau^\alpha(E)}+\|A\psi\|_E+\|A\phi\|_E\big]\end{aligned}$$

does not hold for the solutions of the DSs (3) and (4).

However, the following two results can be established.

Theorem 3. *Assume that A is a strongly PO in a Banach space E and $A\phi-f_1+f_{N-1}$, $A\psi\in E'_\alpha$. Consequently, the solutions of the DSs* (3) *and* (4) *in $\mathbb{C}_\tau^\alpha(E)$ satisfy the following coercive SE*:

$$\begin{aligned}&\big\|\big\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\big\}_1^{N-1}\big\|_{\mathbb{C}_\tau^\alpha(E)}\\&\quad+\big\|\big\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\big\}_1^{N-1}\big\|_{\mathbb{C}_\tau(E'_\alpha)}+\big\|\{Av_k\}_1^{N-1}\big\|_{\mathbb{C}_\tau^\alpha(E)}\\&\leqslant M\alpha^{-1}(1-\alpha)^{-1}\|f^\tau\|_{\mathbb{C}_\tau^\alpha(E)}+M\big(\|A\phi-f_1+f_{N-1}\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}\big),\end{aligned}$$

where M is independent of f^τ, ψ, ϕ, α, and τ.

Proof. By Theorem 1 we have the following estimate for solutions of problems (3) and (4):

$$\begin{aligned}&\big\|\big\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\big\}_1^{N-1}\big\|_{C_\tau^\alpha(E)}+\big\|\{Av_k\}_1^{N-1}\big\|_{C_\tau^\alpha(E)}\\&\quad+\big\|\big\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\big\}_1^{N-1}\big\|_{C_\tau(E'_\alpha)}\\&\leqslant\frac{M}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau^\alpha(E)}+M\big[\|Av_0-f_1\|_{E'_\alpha}+\|Av_N-f_{N-1}\|_{E'_\alpha}\big].\end{aligned}$$

Using nonlocal condition (10), we have that

$$Av_0-f_1=A\phi-f_1+f_{N-1}+Av_N-f_{N-1}.$$

Therefore, to prove the theorem, it suffices to establish the estimate

$$\|Av_N-f_{N-1}\|_{E'_\alpha}\leqslant\frac{M}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau^\alpha(E)}+M\|A\phi-f_1+f_{N-1}\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}.$$

The proof of estimate (13) is based on the triangle inequality, estimates (8) and (9), the definition of the space E'_α, and the following representations of Av_N-f_{N-1} for the solutions of DSs (3) and (4), respectively:

$$\begin{aligned}&Av_N-f_{N-1}\\&\quad=B\big(\mu+(1+\mu\tau)B\big)^{-1}\big(1-Q(\tau B)\big)^{-1}\\&\qquad\times\Bigg\{\big(I-R^N\big)^{-1}\Bigg\{R^N(2I+\tau B)(A\varphi-f_1+f_{N-1})+R^{2N-1}(f_1-f_{N-1})\end{aligned}$$

$$+R^{N-1}(f_1-f_{N-1})+\sum_{i=1}^{N-1}BR^{N-i}(f_i-f_{N-1})\tau-\sum_{i=1}^{N-1}BR^{N+i}(f_i-f_1)\tau\Bigg\}$$
$$-B^{-1}\mu\big[R^{-1}+R^{N-1}\big]f_{N-1}+\big(I+R^N\big)R^{-1}A\psi\Bigg\}$$

and

$$\begin{aligned}
&Av_N-f_{N-1}\\
&\quad=B\big(2(B+\mu)+4\mu\tau B+(3+2\mu\tau)\tau B^2\big)^{-1}\big(1-P(\tau B)\big)^{-1}\\
&\qquad\times\Bigg\{\big(I-R^N\big)^{-1}\Bigg\{(2I+\tau B)\big(-2-2\tau B+\tau^2B^2\big)R^NA\varphi\\
&\qquad-4Bf_{N-1}\tau+B\big(R^2-R^{2N-2}\big)f_{N-1}\tau\\
&\qquad+\big(-3I+R^2\big)\big(I-R^{N-2}\big)f_{N-1}-R^{N-1}\big(-3I+R^2\big)\big(I-R^{N-2}\big)f_1\\
&\qquad+B\big(-3I+R^2\big)\sum_{i=1}^{N-2}R^{N-1+i}(f_i-f_1)\tau\\
&\qquad-\big[2+3\tau B+(-2+\tau B)R^N+\mu B^{-1}\big(2+4\tau B+2\tau^2B^2+2R^N\big)\big]f_{N-1}\\
&\qquad+\big(-3I+R^2\big)B\sum_{i=1}^{N-2}R^{N-1-i}(f_i-f_{N-1})\tau\Bigg\}+2B^{-1}\big(I+R^N\big)R^{-2}A\psi\Bigg\}.
\end{aligned}$$

Therefore, Theorem 3 is proved. □

Theorem 4. *Suppose that A is a strongly PO in a Banach space E and $A\psi$, $A\phi\in E'_\alpha$. Consequently, the solutions of DSs* (3) *and* (4) *in $\mathbb{C}_\tau(E'_\alpha)$ obey the coercive SE*

$$\begin{aligned}
&\big\|\{\tau^{-2}(v_{k+1}-2v_k+v_{k-1})\}_1^{N-1}\big\|_{\mathbb{C}_\tau(E'_\alpha)}+\big\|\{Av_k\}_1^{N-1}\big\|_{\mathbb{C}_\tau(E'_\alpha)}\\
&\qquad\leqslant M\big(\alpha^{-1}(1-\alpha)^{-1}\|f^\tau\|_{\mathbb{C}_\tau(E'_\alpha)}+\|A\phi\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}\big),
\end{aligned}$$

where M does not depend on f^τ, ψ, ϕ, α, and τ.

Proof. By Theorem 2, we have the following estimate for the solutions of problems (3) and (4):

$$\big\|\{Av_k\}_1^{N-1}\big\|_{C_\tau^\alpha(E)}\leqslant M\left[\frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau(E'_\alpha)}+\|Av_0\|_{E'_\alpha}+\|Av_N\|_{E'_\alpha}\right].$$

Using nonlocal condition (10), we have that $Av_0=Av_N+A\phi$. Therefore, in view of the nonlocal condition (10), it suffices to establish the estimate for $\|Av_N\|_{E'_\alpha}$:

$$\|Av_N\|_{E'_\alpha}\leqslant M\left\{\|A\phi\|_{E'_\alpha}+\|A\psi\|_{E'_\alpha}+\frac{1}{\alpha(1-\alpha)}\|f^\tau\|_{C_\tau(E'_\alpha)}\right\}.\tag{14}$$

The proof of estimate (14) is based on estimates (7), (8), and (9), triangle inequality, the definition of the space E'_α, and formulas (12) and (13).

Theorem 4 is proved. □

3 Applications to NBVPs

We now illustrate the use of Theorems 3 and 4 in the analysis of EEs.

First, we consider the NBVP for the two-dimensional EEs with involution in x

$$\begin{aligned}
&-\frac{\partial^2 v(s,x)}{\partial s^2}-\frac{\partial}{\partial x}\left(a(x)\frac{\partial v(s,x)}{\partial x}\right)+\delta v(s,x)-\beta\left(\frac{\partial}{\partial x}\left(a(-x)\frac{\partial v(s,-x)}{\partial x}\right)\right)\\
&\quad=f(s,x),\quad 0<s<T,\ x\in(-l,l),\\
&v(0,x)=v(T,x)+\phi(x),\quad v_s(T,x)+\mu v(T,x)=\psi(x),\quad x\in[-l,l],\\
&v(s,-l)=0,\quad v(s,-l)=0,\quad 0\leqslant s\leqslant T,
\end{aligned}\tag{15}$$

where $\phi(x)$, $\psi(x)$, $a(x)$, and $f(s,x)$ are known smooth functions satisfying the compatibility assumptions that guarantee a smooth solution $v(s,x)$ to problem (15). Besides this, assume that $a\geqslant a(x)=a(-x)\geqslant\sigma>0$, $\sigma-a|\beta|\geqslant 0$, and δ is a sufficiently large positive quantity.

We introduce the differential operator A associated with the NBVP (15) by

$$Au=-\big(a(x)\big(u_x(x)\big)_x+\delta u(x)-\beta\big(a(-x)u_x(-x)\big)_x$$

with the domain $D(A)=\{u\in W_2^2\colon\ u(-l)=0,\ u(l)=0\}$. In [6], the positivity and self-adjointness of the operator A in the Hilbert space $L_2(-l,l)$ were established.

The discretization of NBVP (15) proceeds through two successive steps.

In the first step, we introduce the grid $[-l,l]_h=\{x_j=jh,\ -M\leqslant j\leqslant M,\ Mh=l\}$ and define the Banach space $L_{2h}=L_2[-l,l]_h$ of grid functions $v^h(x)=\{v_j\}_{j=-M}^{M}$ on $[-l,l]_h$ endowed with the norm $\|v^h\|_{L_{2h}}=(\sum_{j=-M}^{M}|v_j|^2h)^{1/2}$. Corresponding to the differential operator A given in (15), we define the difference operator A_h as follows:

$$A_hv^h(x)=\big\{-h^{-2}\big(a(x_{j+1})(v_{j+1}-v_j)-a(x_j)(v_j-v_{j-1})\big)+\delta v_j\big\}_{j=-M+1}^{M-1},$$

acting on grid functions $v^h(x)=\{v_j\}_{j=-M}^{M}$ defined on $[0,l]_h$ that fulfill the relations $v_{-M}=0$ and $v_M=0$.

Using the operator A_h, we obtain the corresponding NBVP

$$\begin{aligned}
&-\frac{\mathrm{d}^2v^h(s,x)}{\mathrm{d}s^2}+A_hv^h(s,x)-\beta A_hv^h(s,-x)=f^h(s,x),\quad x\in[-l,l]_h,\ 0<s<T,\\
&v^h(0,x)=v^h(T,x)+\phi^h(x),\quad v_s^h(T,x)+\mu v^h(T,x)=\psi^h(x),\quad x\in[-l,l]_h.
\end{aligned}\tag{16}$$

In the second stage, problem is replaced by (16) with the abstract DSs (3) and (4), thereby obtaining first- and second-order accurate DSs

$$\begin{aligned}
&-\frac{1}{\tau^2}\big(v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)\big)+A_hv_k^h(x)-\beta A_hv_k^h(-x)=f_k^h(x),\\
&f_kh(x)=f^h(s_k,x),\quad s_k=k\tau\in[0,T]_\tau,\\
&v_0^h(x)=v_N^h(x)+\phi^h(x),\quad (1+\mu\tau)v_N^h(x)-v_{N-1}^h(x)=\tau\psi^h(x),\quad x\in[-l,l]_h,
\end{aligned}\tag{17}$$

and

$$\begin{aligned}
&-\frac{1}{\tau^2}\bigl(v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)\bigr)+A_hv_k^h(x)-\beta A_hv_k^h(-x)=f_k^h(x),\\
&f_k^h(x)=f^h(s_k,x),\quad s_k=k\tau\in[0,T]_\tau,\\
&v_0^h(x)=v_N^h(x)+\phi^h(x),\\
&(3+2\tau\mu)v_N^h(x)-4v_{N-1}^h(x)+v_{N-2}^h(x)=2\tau\psi^h(x),\quad x\in[-l,l]_h,
\end{aligned}\tag{18}$$

for the approximate solution of NBVP (15), respectively.

As a consequence of Theorems 3 and 4, the following well-posedness results for these DSs are obtained.

Theorem 5. *Let h and τ be sufficiently small parameters. Assume that $\psi^h(x)=0$ and $-A_h\phi^h(x)-f^h(0,x)+f^h(T,x)=0$, $x\in[-l,l]_h$. Then the solutions of DSs* (17) *and* (18) *admit the following estimate*:

$$\begin{aligned}
&\bigl\|\bigl\{\tau^{-2}\bigl(v_{k+1}^h-2v_k^h+v_{k-1}^h\bigr)\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(L_{2h})}+\bigl\|\bigl\{v_k^h\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(W_{2h}^2)}+\bigl\|\bigl\{v_k^h\bigr\}_0^{N}\bigr\|_{\mathbb{C}_\tau(W_{2h}^2)}\\
&\quad\leqslant M_{\alpha,\delta,\mu}\bigl\|\bigl\{f_k^h\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(L_{2h})},\quad 0<\alpha<\tfrac{1}{2}.
\end{aligned}$$

Here $M_{\alpha,\delta,\mu}$ does not depend on τ, h, $\{f_k^h\}_1^{N-1}$, ϕ^h, and ψ^h.

Proof. Since A_h is a PO in L_{2h}, DSs (17) and (18) can be replaced by DSs (3) and (4). The proof of Theorem 5 relies on Theorems 3 and 4, as well as on the self-adjointness and positive definiteness of the operator A_h in L_{2h}; see [6].

Second, let Ω be the unit open cube in $\mathbb{R}^n$, given by $0<x_k<1$, $1\leqslant k\leqslant n$, with boundary S, so that $\overline{\Omega}=\Omega\cup S$. On the domain $[0,T]\times\Omega$, we construct DSs for approximating the mixed BVP for a multidimensional EE

$$\begin{aligned}
&-\frac{\partial^2v(s,x)}{\partial s^2}-\sum_{r=1}^n\alpha_r(x)\frac{\partial^2v(s,x)}{\partial x_r^2}+\delta v(s,x)=f(s,x),\\
&\quad 0<y<T,\ x=(x_1,x_2,\ldots,x_n)\in\Omega,\\
&v(0,x)=v(T,x)+\phi(x),\quad v_s(T,x)+\mu v(T,x)=\psi(x),\quad x\in\overline{\Omega},\\
&v(s,x)=0,\quad x\in S,
\end{aligned}\tag{19}$$

where $\alpha_r(x)$, $\phi(x)$, $\psi(x)$, and $f(s,x)$ are known smooth functions satisfying the compatibility assumptions that guarantee a smooth solution $v(s,x)$ to problem (19). Suppose that $\alpha_r(x)\geqslant\alpha_0>0$ and $\delta>0$ is a sufficiently large number.

We introduce the differential operator A associated with the NBVP (19) by

$$Au=-\sum_{r=1}^n\alpha_r(x)\frac{\partial^2u(x)}{\partial x_r^2}+\delta u(x)$$

with the domain $D(A)=\{u(x)\colon\ u(x),u_{x_r}(x),\alpha_r(x)u_{x_rx_r}(x)\in L_2(\Omega),\ 1\leqslant r\leqslant n,\ x\in S\}$. It was established in [7] that the operator A is positive and self-adjoint in the Hilbert space $L_2(\Omega)$.

In the same manner, NBVP (19) is discretized through a two-step procedure.
In the first step, we introduce the corresponding grid sets

$$\begin{aligned}\overline{\Omega}_h &= \big\{x = x_q = (h_1 q_1, \dots, h_m q_m),\ j = (q_1, \dots, q_m),\ 0 \leqslant q_r \leqslant M_r,\\ &\qquad h_r M_r = 1,\ r = 1, \dots, m\big\},\\ \Omega_h &= \overline{\Omega}_h \cap \Omega, \qquad S_h = \overline{\Omega}_h \cap S\end{aligned}$$

and the Banach spaces L_{2h}, $C^{\beta}_{01}(\overline{\Omega}_h)$, and $\mathbb{C}_h = \mathbb{C}(\overline{\Omega}_h)$ of grid functions $v^h(x) = \{v(h_1 q_1, \dots, h_m q_m)\}$ defined on $\overline{\Omega}_h$, endowed with the corresponding norms

$$\big\|v^h\big\|_{L_{2h}} = \Bigg(\sum_{x \in \overline{\Omega}_h} \big|v^h(x)\big|^2 h_1 \cdots h_m\Bigg)^{1/2}$$

and

$$\begin{aligned}\big\|v^h\big\|_{\mathbb{C}^{\beta}_{01}(\overline{\Omega}_h)} &= \big\|v^h\big\|_{\mathbb{C}(\overline{\Omega}_h)}\\ &\quad + \sup_{\substack{0 \leqslant x_k < x_k + h_k \leqslant 1\\ 1 \leqslant k \leqslant m}} \big|v^h(x_1, \dots, x_n) - v^h(x_1 + h_1, \dots, x_m + h_m)\big|\\ &\quad \times h_k^{-\beta_k} \prod_{k=1}^{m} w^{k, M_k}(x, h),\end{aligned}$$

where $w^{k,M_k}(x,h) = x_k^{\beta_k}(1 - x_k - h_k)^{\beta_k}$ and $\|v^h\|_{\mathbb{C}_h} = \sup_{x \in \overline{\Omega}_h} |v^h(x)|$. Here $\beta = (\beta_1, \dots, \beta_m)$.

To the operator A generated by NBVP (19), we associate the difference operator A_h defined by

$$A_h v^h(x) = -\sum_{r=1}^{m} \big(a_r(x) v^h_{\overline{x}_r}(x)\big)_{x_r, q_r} + \delta v^h(x),$$

acting in the space of grid functions $v^h(x)$ satisfying the condition $v^h(x) = 0$ for all $x \in S_h$. The operator A_h is a SAPDO in $L_2(\overline{\Omega}_h)$ and PO in $\mathbb{C}(\overline{\Omega}_h)$); see [7].

By using A_h, we arrive at the NBVP

$$\begin{aligned}&-\frac{\mathrm{d}^2 v(s,x)}{\mathrm{d}s^2} + A_h v^h(s,x) = f^h(s,x), \quad x \in \Omega_h, \quad 0 < s < T,\\ &v^h(0,x) = v^h(T,x) + \phi^h(x), \quad v^h_s(T,x) + \mu v^h(T,x) = \psi^h(x), \quad x \in \overline{\Omega}_h.\end{aligned} \tag{20}$$

During the second step, problem (20) is replaced with abstract DSs (3) and (4). Consequently, we get first- and second-order accurate DSs for the approximating NBVP (19):

$$\begin{aligned}&-\frac{1}{\tau^2}\big(v^h_{k+1}(x) - 2v^h_k(x) + v^h_{k-1}(x)\big) + A_h v^h_k(x) = f^h_k(x),\\ &f^h_k(x) = f^h(s_k, x), \quad s_k = k\tau \in [0,T]_\tau,\ x \in \Omega_h,\\ &v^h_0(x) = v^h_N(x) + \phi^h(x), \quad (1 + \mu\tau)v^h_N(x) - v^h_{N-1}(x) = \tau\psi^h(x), \quad x \in \overline{\Omega}_h,\end{aligned} \tag{21}$$

and

$$
\begin{aligned}
&-\frac{1}{\tau^2}\bigl(v_{k+1}^h(x)-2v_k^h(x)+v_{k-1}^h(x)\bigr)+A_hv_k^h(x)=f_kh(x),\quad y_k\in[0,T]_\tau,\\
&v_0^h(x)=v_N^h(x)+\phi^h(x),\\
&(3+2\tau\mu)v_N^h(x)-4v_{N-1}^h(x)+v_{N-2}^h(x)=2\tau\psi^h(x),\quad x\in\overline{\Omega}_h.
\end{aligned}
\tag{22}
$$

This completes the proof of Theorem 5. □

The following well-posedness results for these DSs follow from Theorems 3 and 4.

Theorem 6. *Let τ and $|h|$ be sufficiently small numbers. Suppose that $\psi^h(x)=0$ and $-A_h\phi^h(x)-f^h(0,x)+f^h(T,x)=0$, $x\in\overline{\Omega}_h$. Then the solutions of DSs* (21) *and* (22) *satisfy the following estimates*:

$$
\begin{aligned}
&\bigl\|\bigl\{\tau^{-2}\bigl(v_{k+1}^h-2v_kh+v_{k-1}^h\bigr)\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(L_{2h})}+\bigl\|\bigl\{v_k^h\bigr\}_0^N\bigr\|_{\mathbb{C}_\tau^\alpha(W_{2h}^2)}+\bigl\|\bigl\{v_k^h\bigr\}_0^N\bigr\|_{\mathbb{C}_\tau(W_{2h}^2)}\\
&\quad\leqslant M_{\alpha,\delta,\mu}\bigl\|\{f_kh\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(L_{2h})},\\
&\bigl\|\bigl\{\tau^{-2}\bigl(v_{k+1}^h-2v_kh+v_{k-1}^h\bigr)\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_h)}+\bigl\|\bigl\{v_k^h\bigr\}_0^N\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_h^2)}+\bigl\|\bigl\{v_k^h\bigr\}_0^N\bigr\|_{\mathbb{C}_\tau(\mathbb{C}_h^2)}\\
&\quad\leqslant M_{\alpha,\delta,\mu}\bigl\|\{f_kh\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_h)},\quad 0<\alpha<\tfrac{1}{2},\\
&\bigl\|\bigl\{\tau^{-2}\bigl(v_{k+1}^h-2v_kh+v_{k-1}^h\bigr)\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_{01}^\beta(\overline{\Omega}_h))}+\bigl\|\bigl\{v_k^h\bigr\}_0^N\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_{01}^{2+\beta}(\overline{\Omega}_h))}\\
&\quad\leqslant M_{\alpha,\delta,\beta,\mu}\bigl\|f_kh\bigr\}_1^{N-1}\bigr\|_{\mathbb{C}_\tau^\alpha(\mathbb{C}_{01}^\beta(\overline{\Omega}_h))},\quad 0<\alpha<\tfrac{1}{2},\quad 0<|\beta_k|<1,\ 1\leqslant k\leqslant m.
\end{aligned}
$$

Here $M_{\alpha,\delta,\mu}$ and $M_{\alpha,\delta,\beta,\mu}$ are independent of τ, h, $\{f_kh\}_1^{N-1}$, ϕ^h, and ψ^h.

Proof. The operator A_h is a strongly PO in $\mathbb{C}_h$, $\mathbb{C}_{01}^\beta(\overline{\Omega}_h)$, and L_{2h}; see [30]. Therefore, DSs (21) and (22) can be replaced by DSs (3) and (4). Consequently, the proof of Theorem 6 relies on Theorems 3 and 4, as well as self-adjointness and positive definiteness of the operator A_h in L_{2h}, the strong positivity of the operator A_h in $\mathbb{C}_h$, $\mathbb{C}_{01}^\beta(\overline{\Omega}_h)$, and the coercivity of elliptic difference operator A_h in $\mathbb{C}_{01}^\beta(\overline{\Omega}_h)$. □

4 Numerical experiments

This section reports numerical results for 2D and 3D Samarskii–Ionkin-type elliptic NBVPs, together with a brief description of their implementation. The computations are carried out in MATLAB.

4.1 2D test example

For test example, in $[0,1]\times[0,1]$, we consider NBVP for two-dimensional EE

$$
\begin{aligned}
&-v_{ss}(s,y)-v_{xx}(s,y)+v(s,y)=f(s,y),\quad 0<y<1,\ 0<s<1,\\
&v(0,y)-v(1,y)=\phi(y),\quad v_s(1,y)+\mu v(1,y)=\psi(y),\quad 0\leqslant y\leqslant 1,\\
&v(s,0)=0,\quad v(s,1)=0,\quad 0\leqslant s\leqslant 1,
\end{aligned}
\tag{23}
$$

where

$$f(s,y)=\bigl(-\mathrm{e}^{-s}+\bigl(1+\pi^2\bigr)\bigl(1+s+\mathrm{e}^{-s}\bigr)\bigr)\sin(\pi y),\qquad \mu=\tfrac{1}{2},$$
$$\phi(y)=-\mathrm{e}^{-1}\sin(\pi y),\qquad \psi(y)=\bigl(-\tfrac{1}{2}\mathrm{e}^{-1}+2\bigr)\sin(\pi y).$$

It is readily verified that $v(s,y)=(1+s+\mathrm{e}^{-s})\sin(\pi y)$ is an exact solution of NBVP (23).

Let

$$[0,1]_\tau\times[0,1]_h=\{(t_k,y_m)\colon\ t_k=k\tau,\ k=\overline{0,N},\ y_m=mh,\ m=\overline{0,M}\},$$

be the rectangular mesh, where $N\tau=1$ and $Mh=1$, and let $\phi_m=\phi(y_m)$, $\psi_m=\psi(y_m)$, and $f_m^k=f(t_k,y_m)$ for $m=\overline{0,M}$ and $k=\overline{0,N}$.

For the numerical solution of the NBVP, we consider the following first- and second-order accurate DSs in t:

$$\begin{aligned}&\frac{v_m^{k+1}-2v_m^k+v_m^{k-1}}{\tau^2}+\frac{v_{m+1}^k-2v_m^k+v_{m-1}^k}{h^2}-v_m^k=-f_m^k,\\ &\quad k=\overline{1,N-1},\ m=\overline{1,M-1},\\ &v_0^k=0,\quad v_M^k=0,\quad k=\overline{1,N-1},\\ &v_m^0-v_m^N=\phi_m,\quad (1+\mu\tau)v_m^N-v_m^{N-1}=\psi_m,\quad m=\overline{0,M},\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\frac{v_m^{k+1}-2v_m^k+v_m^{k-1}}{\tau^2}+\frac{v_{m+1}^k-2v_m^k+v_{m-1}^k}{h^2}-v_m^k=-f_m^k,\\ &\quad k=\overline{1,N-1},\ m=\overline{1,M-1},\\ &v_0^k=0,\quad v_M^k=0,\quad k=\overline{1,N-1},\\ &v_m^0-v_m^N=\phi_m,\quad (3+2\mu\tau)v_m^N-4v_m^{N-1}+v_m^{N-2}=2\tau\psi_m,\quad m=\overline{0,M}.\end{aligned}\tag{25}$$

Both DSs are second-order accurate in x.

By using notations

$$v_l=\bigl[v_0^l,\dots,v_M^l\bigr]^t,\quad l=k\mp1,k,\qquad g_k=\bigl[g_0^k,\dots,g_M^k\bigr]^t,\quad k=\overline{1,N-1},$$
$$\phi=\bigl[\phi_0,\dots,\phi_M\bigr]^t,\qquad \psi=\bigl[\psi_0,\dots,\psi_M\bigr]^t,$$

DSs (24) and (25) can be rewritten in matrix form

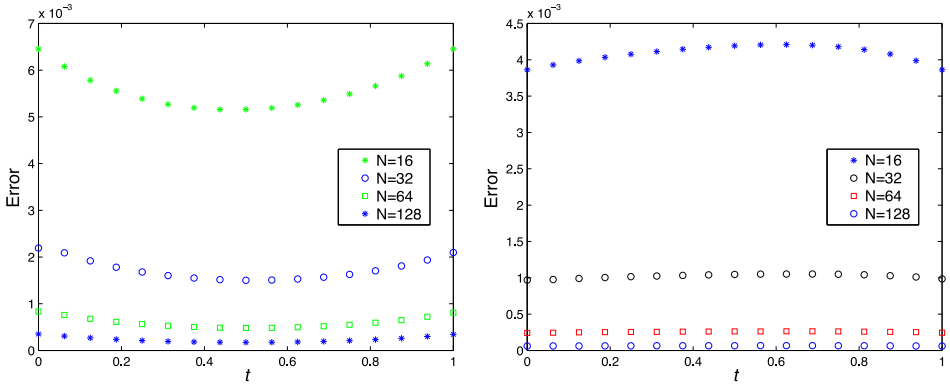
$$\begin{aligned}&Av_{k+1}+Bv_k+Cv_{k-1}=g_k,\quad k=\overline{1,N-1},\\ &v_0=\overrightarrow{0},\qquad v_N=\overrightarrow{0}.\end{aligned}\tag{26}$$

Here $a=1/h^2$, $e=-2/\tau^2-2/h^2-1$, $d=1/\tau^2$, $A=C=\operatorname{diag}(0,a,a,\dots,a,0)$, $g_m^k=-f_m^k$, $m=\overline{1,M-1}$, $k=\overline{1,N-1}$,

$$b_{1,1}=1,\qquad b_{1,N+1}=-1,$$
$$b_{i,i}=e,\quad b_{i-1,i}=d,\quad b_{i,i-1}=d,\quad i=\overline{2,N},$$

Table 1. Error of υ for the test example (23).

(N, M)	First-order accurate DS	Second-order accurate DS
$(2^4, 2^4)$	$3.297 \cdot 10^{-2}$	$3.297 \cdot 10^{-2}$
$(2^5, 2^5)$	$1.250 \cdot 10^{-2}$	$1.250 \cdot 10^{-2}$
$(2^6, 2^6)$	$5.347 \cdot 10^{-3}$	$5.347 \cdot 10^{-3}$
$(2^7, 2^7)$	$2.459 \cdot 10^{-3}$	$2.459 \cdot 10^{-3}$
$(2^8, 2^8)$	$1.178 \cdot 10^{-3}$	$1.178 \cdot 10^{-3}$



(a) First-order accurate DS (b) Second-order accurte DS

Figure 1. Error in the L_{2h}-norm at the same points with different numbers of discretization N.

$$b_{N+1,N-1} = 3 + 2\mu\tau, \qquad b_{N+1,N} = -4, \qquad b_{N+1,N+1} = 1,$$
$$b_{i,j} = 0 \quad \text{for other } i \text{ and } j,\ i, j = \overline{1, N-1},$$

for the first-order accurate DS, and

$$b_{i,i} = e, \quad b_{i-1,i} = d, \quad b_{i,i-1} = d, \quad i = \overline{2, N},$$
$$b_{1,1} = 1, \qquad b_{1,N+1} = -1, \qquad b_{N+1,N} = -1, \qquad b_{N+1,N+1} = 1 + \tfrac{\tau}{2},$$
$$b_{i,j} = 0 \quad \text{for other } i \text{ and } j,\ i, j = \overline{1, N-1}$$

for the second-order accurate DS.

The numerical solution of (26) is obtained using a modified Gauss elimination algorithm. Table 1 contains the results of the numerical experiments carried out for the first- and second-order approximations with $N = M = 2^4, 2^5, 2^6, 2^7$, and 2^8. It displays the differences between the exact solution and the approximate solutions generated by the numerical schemes for the NBVP.

Doubling the number of discretization points results in a twofold reduction in the error of the first-order DS and a fourfold reduction in the error of the second-order DS.

Figures 1(a) and 1(b) display the errors measured in the L_{2h} norm at the same grid points. The errors are computed for the time discretizations $N = 2^4, 2^5, 2^6, 2^7$, and the figures are constructed based on these data. Table 1, together with Figs. 1(a) and 1(b),

demonstrates that the orders of convergence are one and two, respectively. These results confirm the stability of the solutions obtained by both proposed schemes.

4.2 3D test example

We now turn to the three-dimensional elliptic NBVP

$$\begin{aligned}
&-\frac{\partial^2 v}{\partial s^2}(s,y,z)-\frac{\partial^2 v}{\partial x^2}(s,y,z)-\frac{\partial^2 v}{\partial y^2}(s,y,z)+v(s,y,z)\\
&\quad=f(s,y,z),\quad 0<y,z,s<1,\\
&v(0,y,z)-v(1,y,z)=\phi(y,z),\quad 0\leqslant y,z\leqslant 1,\\
&u_s(1,y,z)+\mu u(1,y,z)=\psi(y,z),\quad 0\leqslant y,z\leqslant 1,\\
&v(s,y,0)=0, v(s,y,1)=0,\quad 0\leqslant y,s\leqslant 1,\\
&v(s,0,z)=0, v(s,1,z)=0,\quad 0\leqslant z,s\leqslant 1,
\end{aligned}\tag{27}$$

where

$$\begin{aligned}
f(s,y,z)&=\big(\mathrm{e}^{-s}+\big(2\pi^2+1\big)\big(1+s+\mathrm{e}^{-s}\big)\big)\xi(y,z),\qquad \phi(y,z)=\mathrm{e}^{-1}\xi(y,z),\\
\psi(y,z)&=\big(\mathrm{e}^{-1}(\mu-1)+(1+2\mu)\big)\xi(y,z),\qquad \xi(y,z)=\sin(\pi y)\sin(\pi z).
\end{aligned}$$

It is readily verified that $v(s,y,z)=(1+s+\mathrm{e}^{-s})\xi(y,z)$ is an exact solution of NBVP (27).

Let

$$[0,1]_\tau\times[0,1]_h\times[0,1]_h=\big\{(s_k,y_m,z_n)\colon\ s_k=k\tau,\ k=\overline{0,N},\ y_m=mh,\ z_n=nh,\ m,n=\overline{0,M},\ N\tau=1,\ Mh=1\big\},$$

and let $\phi_{m,n}=\phi(y_m,z_n)$, $\psi_{m,n}=\psi(y_m,z_n)$, and $f^k_{m,n}=f(t_k,y_m,z_n)$ for $k=\overline{0,N}$ and $m,n=\overline{0,M}$. In the first stage, DSs for approximately solution of NBVP can be written in the following forms:

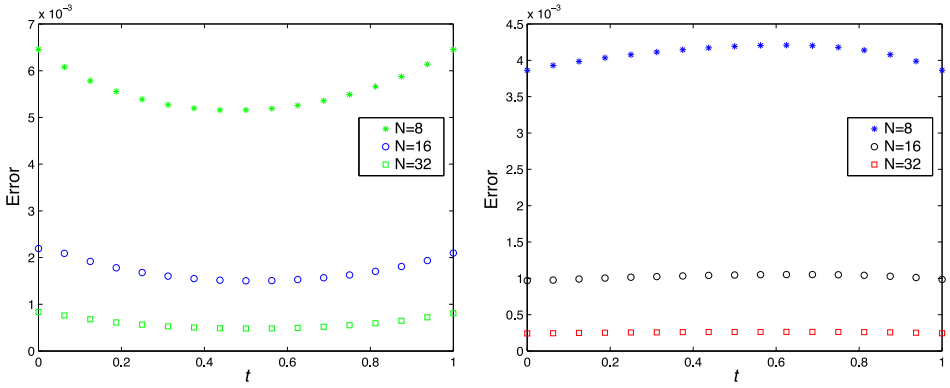
$$\begin{aligned}
&-\frac{v^{k+1}_{m,n}-2v^k_{m,n}+v^{k-1}_{m,n}}{\tau^2}-\frac{v^k_{m,n+1}-2v^k_{m,n}+v^k_{m,n-1}}{h^2}-\frac{v^k_{m+1,n}-2v^k_{m,n}+v^k_{m-1,n}}{\tau^2}\\
&\quad+v^k_{m,n}=f^k_{m,n},\quad k=\overline{1,N-1},\ m,n=\overline{1,M-1},\\
&v^k_{0,n}=0,\quad v^k_{M,n}=0,\quad v^k_{m,n}=0,\quad v^k_{m,M}=0,\quad k=\overline{0,N},\ m,n=\overline{1,M-1},\\
&v^1_{m,n}-v^0_{m,n}=\phi_{m,n},\quad \left(1+\frac{1}{2}\tau\right)v^N_{m,n}-v^{N-1}_{m,n}=\psi_{m,n},\quad m,n=\overline{1,M-1},
\end{aligned}\tag{28}$$

and

$$\begin{aligned}
&-\frac{v^{k+1}_{m,n}-2v^k_{m,n}+v^{k-1}_{m,n}}{\tau^2}-\frac{v^k_{m,n+1}-2v^k_{m,n}+v^k_{m,n-1}}{h^2}-\frac{v^k_{m+1,n}-2v^k_{m,n}+v^k_{m-1,n}}{\tau^2}\\
&\quad+v^k_{m,n}=f^k_{m,n},\quad k=\overline{1,N-1},\ m,n=\overline{1,M-1},\\
&v^k_{0,n}=0,\quad v^k_{M,n}=0,\quad v^k_{m,n}=0,\quad v^k_{m,M}=0,\quad k=\overline{0,N},\ m,n=\overline{1,M-1},\\
&v^1_{m,n}-v^0_{m,n}=\phi_{m,n},\quad m,n=\overline{1,M-1},\\
&(3+2\mu\tau)v^N_{m,n}-4v^{N-1}_{m,n}+v^{N-2}_{m,n}=2\tau\psi_{m,n},\quad m,n=\overline{1,M-1}.
\end{aligned}\tag{29}$$

Table 2. Error of υ for the test example (27).

(N, M)	First-order accurate DS	Second-order accurate DS
$(2^3, 2^3)$	$1.504 \cdot 10^{-2}$	$1.297 \cdot 10^{-2}$
$(2^4, 2^4)$	$4.388 \cdot 10^{-3}$	$3.245 \cdot 10^{-3}$
$(2^5, 2^5)$	$1.393 \cdot 10^{-3}$	$8.115 \cdot 10^{-4}$



(a) First-order accurate DS (b) Second-order accurate DS

Figure 2. Error in the L_{2h}-norm at the same points with different numbers of discretization N.

DSs (28) and (29) are represented in matrix form (26). Table 2 summarizes the errors of the first- and second-order approximations for $N = M = 2^3, 2^4, 2^5$, comparing numerical and exact solutions of the NBVP.

Doubling the number of discretization points results in a twofold reduction in the error of the first-order DS and a fourfold reduction in the error of the second-order DS. Figures 2(a) and 2(b) illustrate the errors of the first- and second-order approximations for the time discretizations $N = 2^3, 2^4$, and 2^5. The figures confirm the stability results for the solutions obtained by both proposed schemes.

5 Conclusion and future plans

NBVPs with the Samarskii–Ionkin condition arise in various applied models where boundary data involve integral or global constraints. However, the construction and rigorous analysis of accurate and stable numerical methods for such elliptic problems in Banach spaces with a PO remain limited.

The nonlocal nature of the boundary conditions creates additional analytical and numerical difficulties, particularly regarding stability and well-posedness. Therefore, developing first- and second-order accurate DSs and establishing their well-posedness in Banach space is both theoretically significant and practically relevant.

This work addresses these issues and supports the theoretical results with numerical experiments for two- and three-dimensional problems. In the present paper, the first- and second-order accurate DSs for the approximate solution of the NBVP with Samarskii–

Ionkin condition for EEs in a Banach space with a PO are studied. The well-posedness of these DSs in Banach spaces is established. In applications, the stability and almost coercive SEs in maximum norm in one variable for the solutions of DSs for numerical solution of elliptic problems are obtained. The presented method is implemented and tested on two- and three-dimensional nonlocal elliptic problems.

Furthermore, by applying the results of [4, 7], one can investigate uniform two-step DSs and derive asymptotic formulas for the solutions of nonlocal boundary value singularly perturbed problems for linear EEs in a Banach space, where a parameter multiplies the highest-order derivative term. In addition, high-order accurate DSs for NBVPs associated with elliptic PDEs can be constructed and rigorously analyzed.

Conflicts of interest. The authors declare no conflicts of interest.

Acknowledgment. The authors would like to thank the reviewers for their careful reading of the manuscript and for their valuable comments and constructive suggestions, which helped improve the quality of this work.

References

1. A. Ashyralyev, A note on the Bitsadze–Samarskii type nonlocal boundary value problem in a Banach space, *J. Math. Anal. Appl.*, **344**(1):557–573, 2008, https://doi.org/10.1016/j.jmaa.2008.03.040.
2. A. Ashyralyev, D. Agirseven, On convergence of difference schemes for delay parabolic equations, *Comput. Math. Appl.*, **66**(7):1232–1244, 2013, https://doi.org/10.1016/j.camwa.2013.07.022.
3. A. Ashyralyev, A. Al-Hammouri, C. Ashyralyyev, On the absolute stable difference scheme for the space-wise dependent source identification problem for elliptic-telegraph equation, *Numer. Methods Partial Differ. Equations*, **37**(2):962–986, 2021, https://doi.org/10.1002/num.22586.
4. A. Ashyralyev, H.O. Fattorini, On uniform difference-schemes for second-order singular perturbation problems in Banach spaces, *SIAM J. Math. Anal.*, **23**(1):29–54, 1992, https://doi.org/10.1137/0523003.
5. A. Ashyralyev, M.A. Sadybekov, A note on the NBVP with Samarskii–Ionkin condition I for elliptic equations, *Uzb. Math. J.*, **69**(2):38–53, 2025.
6. A. Ashyralyev, A. Sarsanbi, Well-posedness of an elliptic equation with involution, *Electron. J. Differ. Equ.*, **2015**(284):1–10, 2015.
7. A. Ashyralyev, P.E. Sobolevskii, *New Difference Schemes for Partial Differential Equations*, Oper. Theory: Adv. Appl., Vol. 148, Birkhäuser, Basel, 2004, https://doi.org/10.1007/978-3-0348-7922-4.
8. A. Ashyralyev, F.S.O. Tetikoglu, A note on Bitsadze–Samarskii type nonlocal boundary value problems: Well-posedness, *Numer. Funct. Anal. Optim.*, **34**(9):939–975, 2013, https://doi.org/10.1080/01630563.2013.805491.

9. C. Ashyralyyev, A fourth order approximation of the Neumann type overdetermined elliptic problem, *Filomat*, **31**(4):967–980, 2017, https://doi.org/10.2298/FIL1704967A.

10. C. Ashyralyyev, M. Dedeturk, Approximation of the inverse elliptic problem with mixed boundary value conditions and overdetermination, *Bound. Value Probl.*, **2015**(1):51, 2015, https://doi.org/10.1186/s13661-015-0312-x.

11. G. Avalishvili, M. Avalishvili, B. Miara, Nonclassical problem with integral boundary conditions for elliptic system, *Complex Var. Elliptic Equ.*, **63**(6):836–853, 2018, https://doi.org/10.1080/17476933.2017.1413244.

12. A. Bakhit, A. Štikonas, O. Štikonienė, Finite difference scheme for two-dimensional Poisson equation with the multiple integral boundary condition, *Mathematics*, **14**(7):1171, 2026, https://doi.org/10.3390/math14071171.

13. V. Būda, M. Sapagovas, O. Štikonienė, A. Štikonas, M-matrices and discrete problems with nonlocal boundary conditions, *Lith. Math. J.*, **65**(2):179–200, 2025, https://doi.org/10.1007/s10986-025-09676-0.

14. A.A. Dukenbayeva, On a generalized problem of the Samarskii–Ionkin type for the Poisson equation in a ball, *Mat. Zh.*, **18**(1):78–87, 2018 (in Russian).

15. A. Hamad, *Fractional Spaces Generated by Positive Operators with the Nonlocal Condition and Their Applications*, PhD thesis, Yakın Doğu Univ., Lefkoşa, Türkiye, 2019.

16. V.A. Il'in, L.V. Kritskov, Properties of spectral expansions corresponding to nonself-adjoint differential operators, *J. Math. Sci., New York*, **116**(5):3489–3550, 2003, https://doi.org/10.1023/A:1023676422956.

17. N.I. Ionkin, The solution of a certain boundary value problem of the theory of heat conduction with a nonclassical boundary condition, *Differ. Uravn.*, **13**(2):294–304, 1977 (in Russian).

18. N.I. Ionkin, Stability of a problem in heat-transfer theory with a nonclassical boundary condition, *Differ. Equations*, **15**:911–914, 1980.

19. B.J. Kadirkulov, F.X. Begimqulov, On a nonlocal problem of the Bitsadze–Samarskii type for an elliptic equation with degeneration, *Lobachevskii J. Math.*, **46**(1):447–455, 2025, https://doi.org/10.1134/S1995080225010175.

20. M. Kirane, B.T. Torebek, On a nonlocal problem for the Laplace equation in the unit ball with fractional boundary conditions, *Math. Methods Appl. Sci.*, **39**(5):1121–1128, 2016, https://doi.org/10.1002/mma.3559.

21. M. Koshanova, B.Kh. Turmetov, K. Usmanov, On solvability of some boundary value problems for a biharmonic equation with periodic conditions, *Filomat*, **32**(3):939–946, 2018, https://doi.org/10.2298/FIL1803939K.

22. A.I. Kozhanov, Nonlocal problems with generalized Samarskii–Ionkin condition for some classes of nonstationary differential equations, *Dokl. Math.*, **107**(1):40–43, 2023, https://doi.org/10.1134/S1064562423700105.

23. A.I. Kozhanov, A.V. Dyuzheva, Well-posedness of the generalized Samarskii–Ionkin problem for elliptic equations in a cylindrical domain, *Differ. Equations*, **59**:223–235, 2023, https://doi.org/10.1134/S0012266123020073.

24. S.G. Krein, *Linear Differential Equations in Banach Space*, AMS, Providence, RI, 1971.

25. E. Ozturk, *Nonlocal Boundary Value Problems for Elliptic Differential and Difference Equations*, PhD thesis, Uludağ Univ., Bursa, Türkiye, 2013.

26. M.A. Sadybekov, A.A. Dukenbayeva, On boundary value problems of the Samarskii–Ionkin type for the Laplace operator in a ball, *Complex Var. Elliptic Equ.*, **67**(2):369–383, 2022, https://doi.org/10.1080/17476933.2020.1846020.

27. A.A. Samarskii, Some problems in differential equation theory, *Differ. Equations*, **16**:1221–1228, 1981.

28. M. Sapagovas, O. Štikonienė, R. Čiupaila, Ž. Jokšienė, Convergence of iterative methods for elliptic equations with integral boundary conditions, *Electron. J. Differ. Equ.*, **2016**(118), 2016.

29. A.L. Skubachevskii, *Elliptic Functional Differential Equations and Applications*, Oper. Theory: Adv. Appl., Vol. 91, Birkhäuser, Basel, 1997, https://doi.org/10.1007/978-3-0348-9033-5.

30. P.E. Sobolevskii, The theory of semigroups and the stability of difference schemes, in *Operator Theory in Function Spaces, Proceedings of the School, Novosibirsk, 1975*, Nauka, Novosibirsk, 1977, pp. 304–337 (in Russian).

31. O.V. Solonukha, On the solvability of an essentially nonlinear elliptic differential equation with nonlocal boundary conditions, *Comput. Math. Math. Phys.*, **64**(2):285–299, 2024, https://doi.org/10.1134/S096554252402012X.

32. F.S. Tetikoğlu, *Fractional Spaces Generated by Positive Differential and Difference Operators with Periodic Conditions and Their Applications*, PhD thesis, Istanbul Univ., Istanbul, Türkiye, 2015.

33. F.S.O. Tetikoglu, *High Order of Accuracy Difference Schemes for Bitsadze–Samarskii Problems*, PhD thesis, Istanbul Univ., Istanbul, Türkiye, 2012.

34. B.Kh. Turmetov, Generalization of the Robin problem for the Laplace equation, *Differ. Equations*, **55**(9):1134–1142, 2019, https://doi.org/10.1134/S0012266119090095.

35. X.C. Zhu, Y. J. Zhu, Existence and limit behavior of constraint minimizers for elliptic equations with two nonlocal terms, *Electron. Res. Arch.*, **32**(8):4991–5009, 2024, https://doi.org/10.3934/era.2024226.